Hedge Fund Fee Structure and Risk Exposure: Theory and Empirical Evidence

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Abstract

We solve in closed form the optimal investment strategy of an infinitely lived risk neutral hedge fund manager compensated by a management fee and a high water mark (HWM) contract. The fraction of asset under management (AUM) allocated in equity is a convex increasing function of the distance to the HWM as moving away from the HWM is increasingly bad news both for management and incentive fees. This convexity effect is enhanced by the size of the incentive fee rate. The higher the management fee rate, the larger the risk exposure, as the revenue insurance effect gets magnified. Frequently beating by a small amount the HWM is optimal as it mitigates the ratchet feature of the HWM. Data seem to support the theoretical predictions of the model: returns' volatility is strongly related to distance to the HWM: being 20% underwater is associated with an increase of 192 bps in the ex-post returns' volatility. Also consistent, the time elapsed between hits and the extent to which the fund surpasses the HWM both increase with distance to the HWM. An extension shows that a fund termination threat reduces risk taking behavior as the fund drifts away from the HWM, which is consistent with our empirical findings.

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1 Introduction

As of the third quarter of 2016, the hedge fund industry was managing an estimated wealth of $2 98 trillion, a slight decrease compared to the pinnacle of $3.02 trillion reported in December 2014. Hedge funds are exempt from many regulatory rules to which the financial industry in general must abide such as, for the case of the United States, the Investment Company Act of 1940, which is an extensive regulatory code[1]. Increased investment opportunities allow managers to implement more flexible strategies and make full use of their talent to deliver profits, which should entitle them to high rewards. Hedge fund managers' compensation exhibits two key features: a management fee, usually, a fraction of the assets under management (AUM) and a performance-incentive fee, typically, a fraction of the fund profits is paid to the fund manager when profits exceed a target value, the high water mark (HWM). The incentive fee intends to align the interests of managers with those of investors: the HWM aims at ensuring that the fund managers' reward is commensurate to performances while keeping track of the history of the fund profits, more specifically, its all time high value ever reached. It can be adjusted to incorporate a minimum return required on the fund, for instance to account for inflation as well as each time some fund inflows or outflows take place. The HWM is a specific feature of the hedge fund industry[2]: The standard remuneration for hedge funds is so called the "2/20-rule", 2% per year of the AUM[3].

In this paper, we study the optimal investment strategy chosen by an infinite lived risk neutral fund manager who earns a management fee as well as a performance fee as previously described. Our baseline model is essentially an extension of the work of Panageas and Westerfield (2009) that accounts for the impact of a management fee rate. In practice, the management fee plays an important role: it will be very difficult for a fund to operate on a daily basis by only relying on bumpy and infrequent hikes in income earned when the HWM is hit[4]. Our motivation is twofold. First we are interested in the combined effect of a management and a performance fees on the level of risk exposure of the fund, in particular how the latter varies with the distance to the HWM, and, on the size of the incentive fee earned each time the HWM is surpassed. Second, using data from the Hedge Fund Research database of monthly observations of returns of both active and liquidated hedge funds over the 1976-2013 period, we test the empirical implications of our model.

Our main result is to show an increasing convex relationship between risk exposure of the AUM and the distance to the HWM. The farther away the fund drifts from the HWM, the smaller the size

[1] The Security and Exchange Commission (SEC) limits the use of short sales, derivative contracts, asset concentration for mutual funds in an attempt to protect investors from high risk investment strategies.

[2] Eighty four pecent of the funds tracked by the HFR Database have a HWM provision.

[3] The fee structure of the typical fund in our data is 1.479 % / 18.309 % and 20 percent of profits in excess of the HWM. Other common fund fee structures include: "1/50-rule".

[4] Lan, Wang and Yang (2013) point out that management fees contribute to the majority of total management compensation and report calibrations in which three quarters of the fund manager's compensation are due to management fees. Calibrations performed in Goetzmann, Ingersoll and Ross (2003) reveal that total lifetime fees could represent 30 up to percent of the value of the AUM, with nearly two third of the cost being due to the management fee.

of the management fee earned and the present value of the incentive fee, which triggers a rise in risk taking behavior by the manager. The economic magnitude is of the effect in the data is large: being 20% underwater is associated with 192 bps increase in the standard deviation of the next 12-month returns or 16.4%.

Regarding the performance fee rate, we find that it has a negative impact on risk taking as the fund manager looks forward and intends to mitigate the ratchet effect of the HWM. The intuition for this result is straightforward: ceteris paribus, when the incentive fee is large, the fund manager chooses a small step strategy that consists in reducing the fund volatility when approaching the HWM: As a result, the HWM is pushed up by a small amount but more often. As stressed in Panageas and Westerfield (2009), the assumption of an infinite horizon is key. Although we do not find this in the data, model and empirical findings coincide to assert that an increase in the performance rate enhances the positive relationship between risk exposure and distance to the HWM.

Numerical simulations reveal that an increase in the management fee rate triggers a more aggressive investment strategy. The intuition is straightforward: A higher management fee rate allows the fund manager to insure part of her compensation, which fosters a risk seeking behavior. The data corroborates this intuition. Then, we compute the expected time until the HWM is reached as a function of the distance to the HWM. It is found to be increasing, validating the intuition that the farther away of the HWM, the longer it takes to collect the next incentive fee. We find evidence consistent with the implication that an increase in the incentive fee rate leads to a smaller AUM volatility, which lowers the expected time.

The extent to which the fund surpasses its HWM is smaller when the fund is close to the high-water mark, as it will be optimal to beat the high-water mark frequently by a small amount to mitigate the ratchet effect. The data supports this prediction. Now, those considerations become less important as the fund gets farther away, and especially when the management fee rate is higher (insurance) and the performance fee rate is smaller (future cost of surpassing the HWM). We find evidence consistent with the latter implication.

We also analyze the impact of the fee structure on the lifetime compensation of the manager and find a negative impact of both management and incentive fees on the fund manager's welfare as the growth of the AUM is thwarted. We provide a closed form expression for the manager's revenue decomposition. Numerical simulations for the baseline model tend to show that the incentive fee is the main source of revenue for the manager. Our data set indicates that as around 48.5% of the revenue of fund managers come from incentive fees, while for instance in Lan, Wang and Yang (2013) this ratio would be around one third (depending on distance).

The closest paper to our baseline model is Drechsler (2014) in which the fund manager has the option to walk away. A solution is derived by assuming that the management fee is proportional to the HWM rather than being proportional to the value of the AUM. Under such an approximation, the optimal investment strategy has an identical pattern to that in absence of management fee. Our model differs from Dreschler (2014) in several dimensions: first of all, a management fee is seen from the client point of view as a loss, so the HWM is not adjusted downward. Second, we derive an exact analytical solution that allows us to uncover some interesting insights on the effects the option like incentive fee contract on the optimal investment strategy and identify the distinct impacts of the management and performance fees on risk exposure.

At the theoretical level, Goetzmann, Ingersoll and Ross (2003) use a contingent claim approach to derive the implied market value of the lifetime fees earned by a manager who has no discretion on portfolio allocations. They find that a significant proportion of managers compensation can be attributed to the incentive fee, in particular for high volatility asset funds for which high manager skills are required. Janeček and Sîrbu (2011) examine a similar problem while allowing for endogenous withdraws from the fund. Guasoni and Obłój (2015) study the case of a CRRA preference fund manager who maximizes the long term certainty equivalent of the cumulated fees paid by the fund. The fee structure is identical to the one considered in our paper; earned fees are required to be invested in the riskfree money market account. The optimal investment strategy consists in allocating a constant fraction of the AUM in the risky asset, whose level depends on the management fee rate and fund manager's risk aversion[5].

This paper is also related to a fairly recent but growing literature on portfolio allocations under wealth performance relative to an exogenous benchmark such as in Browne (1999) and Tepla (2001) or subject to growth objectives required by the decision maker as in Hellwig (2004). In Carpenter (2000), the fund manager is compensated with a call option on the wealth process with a benchmark index as strike price. As in Ross (2004), the author shows that the option compensation does not necessarily lead to more risk seeking. In a similar setting, Buraschi, Kosowski, and Sritrakul (2014) obtain that investment in the risky asset decreases as the AUM approaches the HWM and exceeds the latter up to an extent after which it starts to increase.

An extension to the baseline model introduces an early termination by the investor should the AUM experience a sufficiently large drawdown, measured as fraction of the HWM[6]. Essentially, the presence of the liquidation floor introduces a put option component into the optimal investment strategy in order to restrain and hedge drawdowns of the AUM. Although significantly more complex than the baseline model, we are still able to solve the problem in closed form. We find that the impact on the optimal investment strategy is significant. The closer the AUM gets to the minimum floor, the higher the fund manager's lifetime risk aversion, which curbs down risk exposure. Our empirical findings are consistent with this latter result: risk in funds with high probability of being liquidated is lower and increases less rapidly with the distance to HWM. Depending of the parameters of the model, the optimal fraction of AUM invested in risky asset is either increasing in wealth or hump shaped. The former pattern always prevails when the management fee is small and the liquidation floor is high. Conversely, we observe the latter pattern for sufficiently large management fee rate and low

[5]In a companion paper, Guasoni and Wang (2015) analyze the optimal investment strategies of a risk averse fund manager in charge of either a mutual fund (no HWM) or a hedge fund (with HWM) who is free to invest her own wealth in equity. Investing a constant fraction of the AUM in the stock is still optimal whereas the fund manager shall invest her cumulative earned fees into the riskfree asset and a constant fraction of the rest of his own asset into the stock.

[6]Grossman and Zhou (1993) argue that when "leverage is used extensively, [...] an essential aspect of the evaluation of investment managers and their strategies is the extent to which large drawdowns occur. It is not unusual for the managers to be fired subsequent to achieving a large drawdown (typically above 25 percent)."

liquidation floor, which indicates that as soon as the termination threat is small enough as the AUM has moved away from the liquidation floor, the convex like feature of the compensation scheme induces the optimal investment strategy to exhibit excess risk taking as in the baseline model.

There exists an extensive empirical literature regarding the interplay between compensation contracts with convex payoffs and risk taking behavior that focuses on hedge funds. Results are not always consistent. Brown, Goetzmann and Park (2001) do not find evidence of excessive risk taking behavior when below the HWM. In fact, they argue that fund managers are mainly concerned about their reputation and future in the industry. Studying returns of more than 900 funds over the period 1988-1995, Ackermann, McEnally and Ravenscraft (1999) report that the fear of excess risk taking behavior triggered by the incentive fee seems unfounded. Nevertheless, the incentive fee is a key variable at explaining a risk-adjusted returns (measured by the Sharpe ratio). They also establish a strong positive link between the management fee and the volatility of returns (agency problem). Elton, Gruber, and Blake (2003) report that mutual funds with incentive-fees raise risk exposure after poor performance. Aragon and Nanda (2012) find that funds that perform poorly in absolute terms, relative to others and relative to their HWM tend to increase risk. The effect is stronger with funds with incentive pay but is missing for funds that have HWM provision. The threat of losing AUMs or being liquidated appears to be relevant and even change the direction of the effects. Agarwal et.al (2002) document a convex flow-performance relation and suggest that, in addition to explicit incentives, managers also face significant implicit incentives to risk taking. Since funds charging higher incentive fees exhibit higher money flows, this would induce those to moderate their risk taking behavior. Buraschi et.al (2012) show that funds that have experienced large deviations from their HWM actually reduce volatility.

In our paper, by studying the data in light of a more structured theoretical model we can uncover the mechanism through which the various fees affect risk taking behavior of fund managers. For instance, we point out the insurance role of the management fee. This also allows to test ancillary implications such as the frequency of the hits, the extent to which the fund surpasses the HWM, among others. We are able to explore in a unified empirical framework the role of distance to HWM, relative and absolute performance, the structure of fees, the frequency and extent of HWM surpasses, and the impact of the threat of liquidation. Our results are robust and to some extent we can accommodate some seemingly inconsistent previous results.

The paper is organized as follows. Section 2 describes the baseline setting and contains a heuristic derivation of the optimal solution and an analysis of its properties. Section 3 presents a verification theorem that formally proves the validity of the heuristic solution. In section 4, we discuss an extension of the baseline model that introduces the possibly of early termination of the fund by investors. Section 5 presents the empirical evidence. Section 6 concludes. All proofs are contained in the appendix.

2 Baseline Model

Time is continuous. An infinitely lived[7] risk neutral hedge fund manager has to optimally allocate the AUM of her fund between a risk-free bond and a risky asset (index) in order to maximize her lifetime compensation.

2.1 Financial Markets

There are two securities available in the financial market:

- a risk-free bond whose price B evolves according to

$$dB_t = \widehat{r} B_t dt,$$

where $\widehat{r}$ is the constant interest rate and,

- a stock index whose price S follows a geometric Brownian motion

$$dS_t = S_t(\widehat{\mu} dt + \widehat{\sigma} dw_t),$$

with $S_0 > 0$, where dw_t is the increment of a standard Wiener process w, $\widehat{\mu}$ is the mean return of the stock index S and $\widehat{\sigma}^2$ is its instantaneous variance. All the stochastic processes considered in the paper are assumed to be adapted on a common filtered probability space whose filtration is the one induced by the observations of w.

Let $\widehat{x}$ and $\widehat{z}$ be respectively the *amount* of dollars invested in the riskless bond B and risky security S, so that the wealth process $\widehat{W}$ is equal to $\widehat{x}+\widehat{z}$. Finally, let π denote the fraction of the AUM invested in the risky asset.

2.1.1 AUM Dynamics and High Watermark

Let $c_I > 0$ denote the (constant) withdraw rate by the investor from the fund. For $\lambda > 0$, define

$$\widehat{M}_t = \sup_{0 \le s \le t} \max\{\widehat{M}_0 e^{(\lambda - c_I)t}, \widehat{W}_s e^{(\lambda - c_I)(t-s)}\}.$$

λ is the (minimum) growth rate of the returns required by the investor. Then, set $z_t \equiv \widehat{z}_t e^{-(\lambda - c_I)t}$, $W_t \equiv \widehat{W}_t e^{-(\lambda - c_I)t}$ and $M_t \equiv \widehat{M}_t e^{-(\lambda - c_I)t}$. Observe that

$$M_t = \sup_{0 \le s \le t} \{M_0, W_s; 0 \le s \le t\},$$

and

$$d\widehat{M}_t = (\lambda - c_I)\widehat{M}_t dt + e^{(\lambda - c_I)t} dM_t.$$

[7]This assumption is key for our results. Panageas and Westerfield (2009) show that within a finite horizon framework the volatility of the fund becomes unbounded as approaching the terminal date.

As long as $dM_t = 0$, the HWM $\widehat{M}$ is growing at rate $\lambda - c_I$. At each period, a management fee is charged that is proportional to the AUM with rate $c_F > 0$; usually c_F is around 2%. Thus, total withdraws from the fund take place at (continuous) rate $c = c_F + c_I > 0$. Whenever $dM_t > 0$, the fund manager earns a performance fee equal to $ke^{(\lambda - c_I)t}dM_t$, with $k \in (0,1)$. In practice, k is close to 20%.

Another important force driving the dynamics of the fund is attracting new money[8]. Following Lan, Wang and Yang (2013), we assume that each time the HWM is hit, this triggers some new money inflows $d\widehat{I}$ that are proportional to the fund profits in excess of the HWM. More specifically, at time t,

$$d\widehat{I}_t = ie^{(\lambda - c_I)t}dM_t,$$

where $i > 0$. Exceeding by far the HWM signals the hedge fund manager's asset management skills. The dynamics of the AUM process are given by

$$d\widehat{W}_t = (\widehat{r} - c)\widehat{W}_t dt + (\widehat{\mu} - \widehat{r})\widehat{z}_t dt + \widehat{\sigma}\widehat{z}_t dw_t - (k - i)e^{(\lambda - c_I)t}dM_t,$$

so that the dynamics of discounted AUM process W are given by

$$dW_t = (r - c_F)W_t dt + (\mu - r)z_t dt + \sigma z_t dw_t - (k - i)dM_t, \tag{1}$$

with $r = \widehat{r} - \lambda$, $\mu = \widehat{\mu} - \lambda$ and $\sigma = \widehat{\sigma}$.

2.2 Hedge Fund Optimization Problem

Given $0 < W_0 \leq M_0$, a risk neutral hedge fund manager maximizes the expected value of her management and performance fees, i.e. her objective function F is given by

$$F(W_0, M_0) = \max_{\widehat{z}} E_0 \left[\int_0^\infty e^{-(\widehat{\theta} + \delta)t} (c_F \widehat{W}_t dt + e^{(\lambda - c_I)t} dM_t) \right]$$

$$d\widehat{W}_t = (\widehat{r} - c)\widehat{W}_t dt + (\widehat{\mu} - r)\widehat{z}_t dt + \widehat{\sigma}\widehat{z}_t dw_t - (k - i)e^{\lambda t}dM_t,$$

or equivalently

$$F(W_0, M_0) = \max_{\pi} E_0 \left[\int_0^\infty e^{-(\theta + \delta)t} (c_F W_t dt + k dM_t) \right] \tag{P}$$

$$\text{s.t. } dW_t = (r - c_F)W_t dt + (\mu - r)\pi_t W_t dt + \sigma \pi_t W_t dw_t - (k - i)dM_t,$$

with $\theta = \widehat{\theta} - \lambda + c_I$ is the (adjusted) manager's subjective time discount rate. We also impose a transversality condition

$$\lim_{T \to \infty} E_t \left[e^{-(\theta + \delta)(T + t)} F\left(W_{t+T}, M_{t+T}\right) \right] = 0. \tag{2}$$

[8] Asset growth remains the focus of a majority of managers, in particular, for mid-size fund managers whereas the largest managers have already their growth strategy in place and prefer to concentrate on talent (EY 2015).

Termination is exogenous and follows a Poisson process with constant intensity δ that is independent of the fund returns. We assume that $\theta + \delta > 0$.

2.2.1 Conditions for a Well-Defined Problem

Let β_1 and β_2 be respectively the positive and negative roots of the quadratic Q with

$$Q(y) = \frac{1}{2}\frac{(\mu - r)^2}{\sigma^2}y^2 + (\theta + \delta - r + c_F - \frac{1}{2}\frac{(\mu - r)^2}{\sigma^2})y - (\theta + \delta),$$

We make the following assumptions:

A1. Growth condition: $\beta_2(k - i) + 1 + i < 0$.

This is a similar condition as in Panageas and Westerfield (2009). It can be seen as a non-Ponzi game or tranversality condition that ensures that $F(W, M) < \infty$ (see Appendix).

A2. $\mu \neq r$.

When $\mu = r$ optimization problem P is ill-posed as the optimal investment strategy π^* is unbounded; for more details, see Panageas and Westerfield (2009).

A3. $r > c_F$.

Condition A3.[9] guaranties that investing all the wealth into the riskless asset continuously (and infinitesimally) increases the HWM. Worth observing is the fact that $r > c_F$ implies that $\beta_1 > 1$.

Finally, whenever $W_t \geq M_t$ we have

$$F(W_t, M_t) = kdM_t + F(W_t + (i - k)dM_t, M_t + (1 + i)dM_t)).$$

Taking a Taylor expansion and letting dM_t goes to zero leads to

$$(k - i)F_1(M_t, M_t) = k + (1 + i)F_2(M_t, M_t).$$

This last condition ensures that the value function is continuous at $W_t = M_t$.

2.2.2 Primal Value Function

Due to the homogeneity of degree 1 of the hedge fund manager' compensation function and the wealth dynamics equation (1), the value function F is homogeneous of degree 1 so we can write

$$F(W, M) = Mf(u),$$

where $u = \frac{W}{M}$, for some smooth function f. In the rest of the paper, we shall refer to u as "the distance to the high water mark". Also note that clearly $F_1 \geq 0$, so $f' \geq 0$. The boundary condition at $u = 1$

[9] This condition is not required for the existence of optimization problem P but nevertheless simplifies the analysis and the exposition of the results. In the general case, results can be derived relying on the confluent hypergeometric functions and their properties.

is

$$(1+k)f'(1) = k + (1+i)f(1). \tag{3}$$

For $u < 1$, the reduced Hamilton Jacobi Bellman (HJB) equation satisfied by f is:

$$(\theta+\delta)f(u) = c_F u + (r - c_F)uf'(u) + \max_{\pi} \pi(\mu - r)uf'(u) + \frac{\sigma^2}{2}\pi^2 u^2 f''(u). \tag{4}$$

Assuming that f is a concave function (we prove this claim in the sequel), it follows that

$$\pi^* = -\frac{\mu - r}{\sigma^2}\frac{f'(u)}{uf''(u)},$$

and the reduced HJB is

$$(\theta+\delta)f(u) = c_F u + (r - c_F)uf'(u) - \frac{1}{2}\frac{(\mu-r)^2}{\sigma^2}\frac{(f'(u))^2}{f''(u)}. \tag{5}$$

2.2.3 Dual Value Function

Let function J be a Legendre transform of value function f. Dual variables (u, x) satisfy

$$x = f'(u) \text{ and } u = -J'(x),$$

and $f(u) = J(x) - xJ'(x)$ as well as $J(x) = f(u) - uf'(u)$. Set $\Lambda = -\frac{c_F\beta_1\beta_2}{\theta+\delta} > 0$. The dual (reduced) HJB satisfies:

$$x^2 J''(x) + [(1-\beta_1-\beta_2)x - \Lambda]J'(x) + \beta_1\beta_2 J(x) = 0. \tag{6}$$

The general solution of (6) is given by:

$$J(x) = K_1 H_1(x) + K_2 H_2(x),$$

with

$$\begin{aligned} H_1(x) &= x^{\beta_2}\int_0^\infty e^{-\frac{\Lambda(1+t)}{x}} t^{\beta_1}(1+t)^{-\beta_2-1}dt \\ H_2(x) &= x^{\beta_2}\int_0^1 e^{-\frac{\Lambda t}{x}} t^{-\beta_2-1}(1-t)^{\beta_1}dt. \end{aligned}$$

Some useful properties of functions H_1 and H_2 are provided in the Appendix.

We are looking for a solution of (6) J defined on some interval $I \sqsubseteq \mathbb{R}_+$ such that: (ii) J is non-negative on I, (ii) J' is negative on I, (iii) J'' is positive on I and, (iv) at some extremity of interval I, both J and J' are equal to zero. In addition, at $u = 0$, the boundary condition is $f(u) = 0$.

Proposition 1 *The reduced dual value function J is defined on the interval $[x^*, \infty)$, is decreasing and strictly convex and is given*

$$J(x) = -\frac{H_2(x)}{H_2'(x^*)},$$

where $x^* > 1$ *is uniquely defined by* $(k-i)x^* - k + (1+i)\frac{H_2(x^*)}{H_2'(x^*)} = 0$.

Proof. See the Appendix. ■

Note that the strict convexity of J implies the strict concavity of f, so the interior solution for maximization problem in (4) is justified. The definition of x^* is implied by condition (3). The next proposition summarizes the properties of the reduced value function f.

Proposition 2 *The value function* F *is homogeneous of degree* 1, *strictly increasing in* W *and* M *and strictly concave in* W *and in* M. *For* (c_F, c_I, i) *given, if* $k_1 < k_2$, *then for all* $u \in [0,1]$, *we have* $f_1(u) > f_2(u)$. *For* (c, k) *given, if* $i_1 < i_2$, *then for all* $u \in [0,1]$, *we have* $f_1(u) < f_2(u)$. *For* (k, i, c_I) *given, if* $c_{F_1} < c_{F_2}$, *then for all* $u \in [0,1]$, *we have* $f_1(u) > f_2(u)$. *Finally, a representation of the (optimal) reduced wealth process* $u \in (0,1)$ *is given by*

$$u_t = -J'(x_t),$$

where

$$x_t = x_0 + \int_0^t ((\theta + \delta - r + c_F)x_s - c_F)\, ds - \frac{\mu - r}{\sigma} \int_0^t x_s dw_s, \qquad (7)$$

with $x_0 > x^*$ *satisfying* $u_0 = -\frac{H_2'(x_0)}{H_2'(x^*)}$. *Process* x *is mean reverting if and only if* $\beta_1 + \beta_2 < 1$.

Proof. See the Appendix. ■

The higher either the management or the incentive fee rate, the lower the lifetime manager compensation. A high fee rate reduces the growth of the AUM, in particular in an infinite horizon setting (see Panageas and Westerfield (2009)). Overall, this overcomes the positive effect for the manager of collecting a larger fraction of the AUM as well as a larger fraction of the performance reward. We note that the impact of the management fee rate is in sharp contrast with the benchmark case where there is no HWM. In the presence of a HWM, a higher management fee rate reduces all the more the AUM, making it harder to beat the target.

2.2.4 Baseline Parameter Set

We calibrate the baseline model using the following values for the parameters: $\theta + \delta = 0.16$, $\mu - c_I = 0.07$, $r - c_I = 0.03$, $\sigma = 0.25$, $c_F = 2\%$, $k = 20\%$ and $i = 0\%$. One can check that under this set of parameters, condition A3 is indeed satisfied.

Unless specified otherwise, all the simulations are performed for this set of parameters and we shall investigate the quantitative impact of parameters c_F and k.

We now examine the fund manager compensation decomposition between, on the one hand earned fees for managing the fund and, on the other hand earned fees based on performance.

2.3 Fund Manager Compensation Decomposition

Let

$$\begin{aligned} F_c(W,M) &= E_0\left[\int_0^\infty c_F e^{-(\theta+\delta)t} W_t dt\right] \\ F_k(W,M) &= E_0\left[\int_0^\infty k e^{-(\theta+\delta)t} dM_t\right], \end{aligned}$$

be the lifetime cumulative management and performance fees earned by the fund manager, respectively. By homogeneity, we can write $F_c(W,M) = Mf_c(u)$ and $F_k(W,M) = Mf_k(u)$, with $f_c(0) = f_k(0) = 0$ and note that F_k satisfies the same boundary condition than F at $W = M$. It follows that

$$\begin{aligned} (1+k)f_c'(1) &= (1+i)f_c(1) \\ (1+k)f_k'(1) &= k + (1+i)f_k(1). \end{aligned}$$

Recall that $u = -J'(x)$ and define functions g_k and g_c such that

$$\begin{aligned} g_k(x) &\triangleq f_k(-J'(x)) \\ g_c(x) &\triangleq = f_c(-J'(x)), \end{aligned}$$

For $x > x^*$, g_k satisfies the following HJB

$$(\theta+\delta)g_k(x) = [(\theta+\delta-r+c_F)x - c_F)]g_k'(x) + \frac{1}{2}\frac{(\mu-r)^2}{\sigma^2}x^2 g_k''(x),$$

and note that $\lim_{\infty} g_k = 0$. Similarly, we have

$$(\theta+\delta)g_c(x) = -c_F J'(x) + [(\theta+\delta-r+c_F)x - c_F)]g_c'(x) + \frac{1}{2}\frac{(\mu-r)^2}{\sigma^2}x^2 g_c''(x).$$

Proposition 3 *The lifetime cumulative performance fee F_k earned by the fund manager is increasing and strictly concave in W and is given by*

$$F_k(W,M) = A_k M\left[f(u) - uf'(u)\right],$$

where

$$A_k = \frac{k}{-(k+1)\frac{H_2'(x^*)}{H_2''(x^*)} + (1+i)\frac{H_2(x^*)}{H_2'(x^*)}} > 0.$$

Proof. See the Appendix. ■

Interestingly, the cumulative lifetime performance fee is proportional to the marginal value of the total earned fee with respect to the HWM. Next, we look at the share of the performance fee in the total compensation, i.e. the ratio

$$\frac{f_k(u)}{f(u)} = A_k\left(1 - \frac{uf'(u)}{f(u)}\right).$$

In Appendix D1, we establish that $\lim_{u \to 0} \frac{f_k(u)}{f(u)} = \frac{A_k}{1-\beta_2}$ and $\frac{f_k}{f}$ is increasing in u, which is fairly intuitive as the closer to the HWM, the larger the option value associated with surpassing the HWM. Numerical simulations are performed for the baseline parameter case with an exception for parameter $c_F = 3\%$.

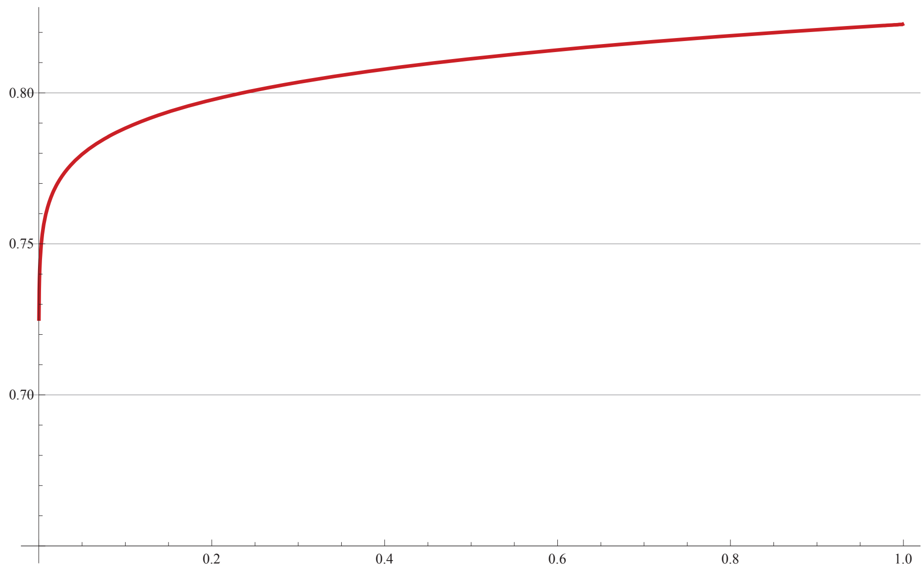

Figure 1 : Share of the performance fee as a function of $\frac{W}{M}$

Figure 1 depicts the share of the total compensation due to the performance fee. Figure 1 reveals that the fund manager derives most of her revenues from the performance fee part and, the closer to the HWM, the larger the fraction of the fund manager's compensation due to the performance fee. For instance, at $u = 0.8$, the management fee only accounts for 15% of the total compensation. These results are in sharp contrast with the results obtained in Lan, Wang and Yang (2013) where the fund manager being concerned with downside liquidation risk chooses a (much) more prudent leverage level than in our setting.

To understand the magnitude of the incentive fee, one needs to investigate the mechanism between risk taking and exceeding the HWM, in particular the impact of portfolio holdings in the neighborhood of the HWM. As derived in Grossman and Zhou (1993)

$$E_t \left[M_{t+h} - M_t \mid W_t = M_t \right] = \sqrt{\frac{2}{\pi}} \sigma \left| \pi_t^* \right| M_t \sqrt{h} + O(h). \tag{8}$$

The expected increase in the HWM over the time interval $[t, t+h]$ is proportional to the fraction π^* of the AUM invested in equity and $\sqrt{h}$. As $\sqrt{h}$ dominates h, clearly an increase in the HWM has a significant (instantaneous) impact on the manager's compensation. In section 5, we report that on average, the HWM is surpassed by a margin of 12.7%.

2.4 Optimal Investment Strategy

Proposition 4 *The fraction of AUM invested in the risky asset π^* is decreasing in the ratio $\frac{W}{M}$ with $\pi^* \leq \frac{\mu-r}{\sigma^2}(1-\beta_2)$ and uniformly decreasing (resp. increasing) in k (resp. i).*

Proof. See the Appendix. ■

The interpretation of proposition 4 is quite intuitive. Recall that in the baseline model, there is no penalty for depleting the fund: the deeper out of the money her incentive contract, the higher the risk exposure in order to hasten wealth accumulation. Although the fund manager is assumed to be risk neutral, the ratio $R_R(u) = -\frac{uf''(u)}{f'(u)}$ can be interpreted as a measure of her lifetime relative risk aversion; R_R is increasing in u, i.e. the fund manager's lifetime utility of the manager exhibits increasing relative risk aversion (IRRA). Drechsler (2014) shows that a similar result may arise, eventhough no management fee is charged, when the outside payoff of the fund manager is sufficiently large with respect to the continuition value at the liquidation threshold. In our setting, this is no liquidation threat and excess risk behavior is induced by the presence of the management fee: as further developed in the sequel, numerical simulations show that the larger the management fee rate c_F, the more the optimal investment strategy is tilted towards the risky asset. There is an extensive literature that argues that the convex payoff structure in hedge fund fees creates incentives for the manager to take excess risk and, in particular when the contract reward is deep out of the money. This result is in line with Carpenter (2000) and Ross (2004). Interestingly, the maximum value of the fraction of the AUM invested in the risky asset is bounded and equal to $\frac{\mu-r}{\sigma^2}(1-\beta_2)$. This is the same expression as the optimal constant investment strategy derived in Panageas and Westerfield (2009) when no management fee is charged, even though the level is higher[10]. This indicates that the optimal investment strategy derived in Panageas and Westerfield (2009) does exhibit excess risk behavior; in fact, it corresponds to the case where the HWM is always seen as "infinitely" far away. Nevertheless, the optimal constant fraction of wealth invested in the risky asset is (uniformly) lower as additional excess risk behavior due the management fee does not take place. Finally, note that the minimum value of the fraction of wealth is reached when the HWM is hit and is equal to $-\frac{\mu-r}{\sigma^2}\frac{x^*H_2''(x^*)}{H_2'(x^*)}$. Unless as in Panageas and Westerfield (2009), this ratio depends on the performance fee rate k.

The optimal investment strategy reflects the intertemporal trade-off faced by the manager between (i) her short term objective, namely earning a (high) management fee and beating the high-water mark, and (ii) her long term objective, i.e. the continuation value. To illustrate this intertemporal trade-off, assume that the HWM is surpassed by a margin of q percent. The fund manager only pockets fee kqM, but the AUM level is now $M(1+(1+i-k)q)$ and the new HWM is $M(1+(1+i)q)$. This implies that the AUM will have to grow by $\frac{kq}{1+(1+i-k)q}$ percent to hit again the HWM; this ratio is indeed increasing in q.

We now examine the impact of the fee structure on the optimal investment strategy.

[10]It is easy to show that β_2 is decreasing in c_F.

2.4.1 Impact of the Performance Fee Rate

Proposition 5 *For (c,i) given, if $k_1 < k_2$, then for all $u \in [0,1]$, we have $\pi_1^*(u) \geq \pi_2^*(u)$. For (c,k) given, if $i_1 < i_2$, then for all $u \in [0,1]$, we have $\pi_1^*(u) \leq \pi_2^*(u)$.*

Proof. See the Appendix. ■

The manager has all the more incentives to inflate the fund volatility near the HWM as the performance fee rate k is small. When k is high, the manager incentives to beat the HWM by a large amount are reduced. This reflects the aforementioned intertemporal trade-off faced by the fund manager. As mentioned in Panageas and Westerfield (2009), a contract with a HWM can be seen a sequence of options with changing strike each time the HWM is reset. The optimal strategy consists in often beating the HWM by small amounts rather than to beating the HWM by large amounts infrenquently. Finally, the larger parameter i, the larger the inflow of new money that is, by assumption, proportional to the performance of the fund manager at exceeding the HWM, which provides extra incentives to take risk.

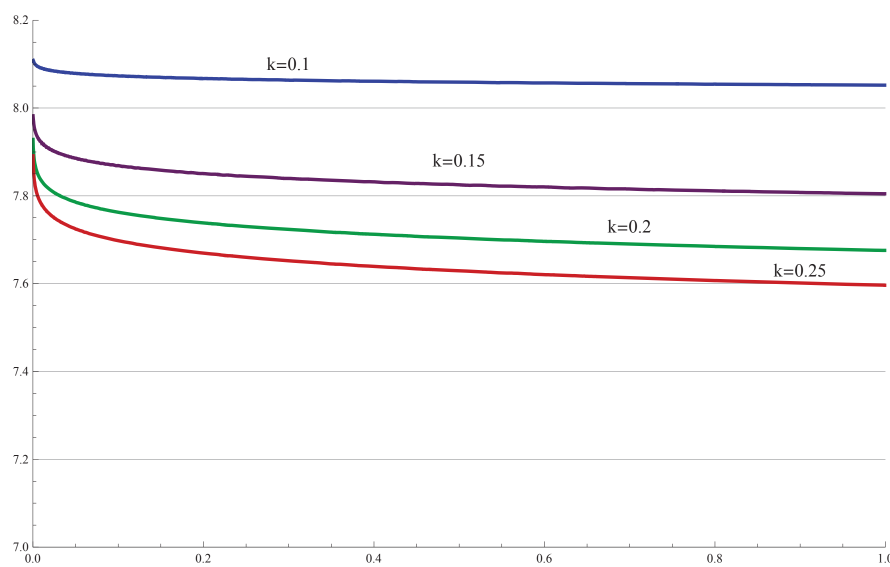

Figure 2 : Fraction of the AUM invested in stocks as a function of $\frac{W}{M}$

Worth mentioning is the fact that the effect of distance to the HWM on the increase in risk is all the more severe as the incentive fee rate gets larger.

2.4.2 Impact of the Management Fee Rate

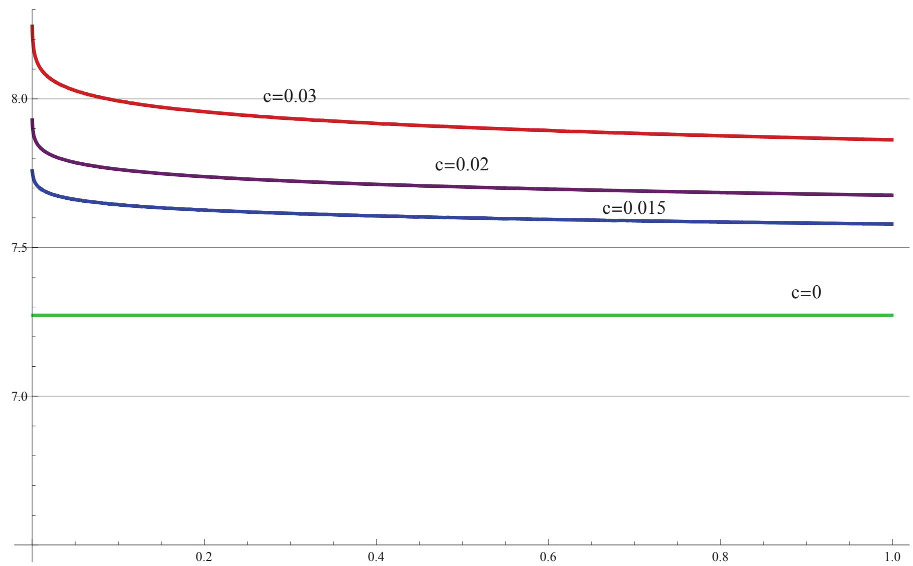

Figure 3 : Fraction of the AUM invested in stocks as a function of $\frac{W}{M}$

Numerical simulations suggest that, the higher the management fee rate, the larger the fund manager's appetite for risk. The management fee acts as an insurance: ceteris paribus, a fund manager earning a hefty management fee is more keen on increasing risk exposure as her revenues are smoothed out across time. The size of the management fee and the present value of the incentive fee shrinks all the more as the fund drifts away from the HWM, which triggers a rise in risk taking behavior by the manager.

2.4.3 How Often is the HWM Hit?

Define the stopping time until next hit

$$\tau = \inf_{t \geq 0} \{u_\tau \geq 1,\ u_0 < 1, \text{ given}\},$$

where $u_0 = -\frac{H_2'(x_0)}{H_2'(x^*)}$. Then, let us introduce the auxiliary function A with

$$A(x) = \int_0^1 e^{-\frac{\Lambda t}{x}} (1-t)^{\beta_1+\beta_2} \left[1 + \left(\frac{\Lambda(1-t)}{x} + \beta_1 + \beta_2 + 1\right) \ln t\right] dt.$$

Finally, we assume that $\beta_1 + \beta_2 > 0$ so that $E[\tau] < \infty$.

Proposition 6 *For an initial condition $u_0 < 1$, the expected time until the HWM is hit is given by*

$$E[\tau] = \frac{1}{(\beta_1 + \beta_2)\frac{1}{2}\frac{(\mu - r)^2}{\sigma^2}} \left(\ln \frac{x_0}{x^*} + A(x^*) - A(x_0) \right),$$

where $u_0 = -\frac{H_2'(x_0)}{H_2'(x^*)}$.

Proof. See the Appendix. ■

First of all, we note that $E[\tau]$ is not always finite, so there may be a positive probability that the HWM is never hit. Condition $\beta_1 + \beta_2 > 0$ and condition A.1 are not always jointly met under the baseline parameter set asit is not easy to have the two conditions satisfied for a reasonable set of parameters. We perform simulations with $\theta + \delta = 0.01$, $\mu = 0.04$, $r = 0.03$, $\sigma = 0.4$, $c_F = 2\%$, $k = 40\%$.

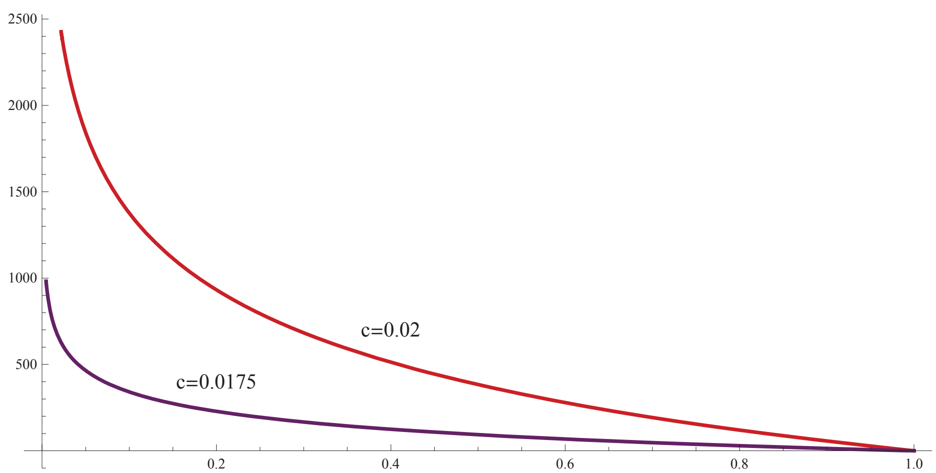

Figure 4 : Expected Time until Hitting the HWM as a function of $\frac{W}{M}$

Figure 4 depicts the expected time until the HWM is hit as a function the distance to the HWM for several values of the management fee rate. We note that the expected time is increasing and convex as a function of the distance to the HWM making the HWM increasingly difficult to surpass as one moves away from it. This result is line with the previously developed intuition. Then, to infer the impact of the management fee on the expected time, we have to investigate two effects (i) the impact of c_F on target level x^*, (ii) the impact of c_F on the law of motion of process x. First of all, recall that we found that $\frac{\partial x^*}{\partial c_F} < 0$. Then, recall that process x remains above $x^* > 1$ and only its drift $\mu_x = (\theta + \delta - r + c_F)x - c_F$ depends on parameter c_F. An increase in c_F raises the drift of x, thwarting any decrease in process x. Therefore, we should expect that the higher the management fee rate, the more infrequently the HWM is hit. Indeed, numerical simulations seem to confirm this intuition.

2.5 Market Value of Earned Fees

So far, we have examined the present value of the fees collected by the fund's manager. We now derive the "fair value" or the market value of a claim whose payoffs are equal to the fees earned, assuming such a claim is marketable. This allows us to compare our results with the paper by Goetzmann, Ingersoll and Ross (2003) but unlike the latter paper, the investment strategy is optimal, in the sense that it maximizes the objective function of the fund manager. One of the main implication is that the volatility of the AUM is no longer constant as in Goetzmann, Ingersoll and Ross (2003) but instead depends on the distance to the HWM.

We assume that there exists a unique state price density[11] ξ whose dynamics are given by

$$d\xi_t = \xi_t(-rdt - \frac{\mu - r}{\sigma}dw_t).$$

In addition, in order to simply the analysis, we set $\delta = 0$. The market value of the cumulative fees earned by the fund manager is given by

$$V(W_0, M_0, \xi_0) = \frac{1}{\xi_0}E_0\left[\int_0^\infty c_F\xi_t W_t dt + k\xi_t dM_t\right].$$

By homogeneity, we have

$$V(W_0, M_0, \xi_0) = M_0 v(u_0).$$

Under the risk neutral probability measure, for all $u < 1$, the return of the AUM must be equal to $r - c_F$. It follows that the market price v must satisfy the following Black-Scholes type PDE for $u < 1$

$$rv(u) = c_F u + (r - c_F)uv'(u) + \frac{\sigma_u^2(u)}{2}u^2 v''(u),$$

where $\sigma_u(u) = -\frac{\mu - r}{\sigma}\frac{f'(u)}{uf''(u)}$ is the volatility of the AUM. From Proposition 2, we can write $v(u_0) = v(-J'(x_0)) \triangleq g(x_0)$. Then, let us write the decomposition of the (market) values of the cumulative management fee and performance fee as

$$\begin{aligned} \xi_0 M_0 \overline{g}_c(x_0) &= E_0\left[\int_0^\infty c_F \xi_t W_t dt\right] \\ \xi_0 M_0 \overline{g}_k(x_0) &= E_0\left[\int_0^\infty k\xi_t dM_t\right]. \end{aligned}$$

As W is marketable, we have

$$E_0\left[\int_0^\infty c_F\xi_t W_t dt + k\xi_t dM_t\right] = \xi_0 W_0 + E_0\left[\int_0^\infty i\xi_t dM_t\right],$$

i.e., the total compensation of the fund manager is equal to the current value of the AUM augmented

[11] To be more precise, the unique state price density is ξ' that follows the same dynamics as ξ for (market) parameters $\widehat{r}$ and $\widehat{\mu}$.

by the cumulated value of the new money flows. This implies that for all $x > x^{**}$

$$-J'(x) = \overline{g}_c(x) + (1 - \frac{i}{k})\overline{g}_k(x),$$

and $\overline{g}_k$ satisfies the following HJB

$$r\overline{g}_k(x) = [(\theta - r + c_F + \frac{(\mu - r)^2}{\sigma^2})x - c_F)]\overline{g}_k'(x) + \frac{1}{2}\frac{(\mu - r)^2}{\sigma^2}x^2\overline{g}_k''(x). \tag{9}$$

Similarly, we have

$$r\overline{g}_c(x) = -c_F J'(x) + [(\theta - r + c_F + \frac{(\mu - r)^2}{\sigma^2})x - c_F)]\overline{g}_c'(x) + \frac{1}{2}\frac{(\mu - r)^2}{\sigma^2}x^2\overline{g}_c''(x).$$

Let α_1 and α_2 be respectively the positive and negative roots of the quadratic $\widehat{Q}$ with

$$\widehat{Q}(y) = \frac{1}{2}\frac{(\mu - r)^2}{\sigma^2}y^2 + (\theta - r + c_F + \frac{1}{2}\frac{(\mu - r)^2}{\sigma^2})y - r.$$

In the sequel, we shall assume that $\alpha_1 > 1$, or equivalently $\frac{(\mu-r)^2}{\sigma^2} + \theta + c < 2r$, so that, an integral representation is available. Two independent solutions to ODE (9) are

$$\begin{aligned} V_1(x) &= x^{\alpha_2}\int_0^\infty e^{-\frac{\Delta(1+t)}{x}}t^{\alpha_1}(1+t)^{-\alpha_2-1}dt \\ V_2(x) &= x^{\alpha_2}\int_0^1 e^{-\frac{\Delta t}{x}}t^{-\alpha_2-1}(1-t)^{\alpha_1}dt, \end{aligned}$$

where $\Delta = -\frac{c_F\alpha_1\alpha_2}{r} > 0$. The general solution to (9) that vanishes as x goes to ∞ is given by

$$\overline{g}_k(x) = B_k V_2(x),$$

where B_k is a constant to be determined. At $x = x^*$, function $\overline{g}_k$ satisfies

$$-(1+k)\frac{\overline{g}_k'(x^*)}{J''(x^*)} = k + (1+i)\overline{g}_k(x^*),$$

and recall that $J''(x^*) = -\frac{H_2''(x^*)}{H_2'(x^*)}$. This leads to

$$B_k = \frac{k}{(1+k)\frac{V_2'(x^*)H_2'(x^*)}{H_2''(x^*)} - (1+i)V_2(x^*)}.$$

The market value of the cumulative management fee is given by

$$\overline{g}_c(x) = \left[\frac{H_2'(x)}{H_2'(x^*)} - (1 - \frac{i}{k})B_k V_2(x)\right].$$

For the baseline set of parameters, numerical simulations (not reported here) reveal that, as in the case of the discounted value of cumulative fee, the fund manager earns most of her remuneration through the performance fee.

3 The Verification Theorem

We present a verification theorem (see for instance Dybvig (1996)) to show that the heuristic proposed optimal strategy π^* is indeed valid. We closely follow Panageas and Westerfield (2009). The proof consists of two steps.

For any feasible strategy π, let define the process

$$Q_t^\pi = \int_0^t e^{-(\theta+\delta)s}(c_F W_s^\pi ds + kdM_s^\pi) + e^{-(\theta+\delta)t} F(W_t^\pi, M_t^\pi),$$

where F is the proposed (optimal) value function. Let $\overline{M} > 0$ and denote $\overline{\tau} = \inf_{t \geq 0} \{M_t \geq \overline{M}\}$.

Step 1: We look for a function $\overline{F}$ such that

$$(\theta+\delta)\overline{F} = c_F W + (r - c_F)W\overline{F}_1 - \frac{1}{2}\frac{(\mu-r)^2}{\sigma^2}\frac{\overline{F}_1^2}{\overline{F}_{11}},$$

that satisfies the boundary conditions

$$\begin{aligned} k\overline{F}_1(M,M) &= k + (1+i)\overline{F}_2(M,M) \\ \overline{F}(W,\overline{M}) &= 0 \text{ for all } 0 < W \leq \overline{M} \\ F(0,M) &= 0 \text{ for all } 0 < M \leq \overline{M}. \end{aligned}$$

Let us consider the following Legendre transform

$$\begin{aligned} W &= -\overline{J}_1(x,M) \\ x &= \overline{F}_1(W,M). \end{aligned}$$

The solution we are looking for is of the form $\overline{J}(x,M) = K(M)H_2(x)$, for some smooth function K. It follows that

$$\overline{F}(W,M) = K(M)\left[H_2(x) - xH_2'(x)\right],$$

As $H_2'' > 0$, by the Implicit Function Theorem, define function X such that

$$M = -\overline{J}_1(X(M),M) = -K(M)H_2'(X(M)).$$

The boundary condition at $W = M$ leads to

$$\begin{aligned}(k-i)X(M) - k &= (1+i)K'(M)H_2(X(M)) \\ M &= -K(M)H_2'(X(M)),\end{aligned}$$

with $K(\overline{M}) = 0$ so that we must have $X(\overline{M}) = 0$. Eliminating function K, we find that X satisfies the following ODE

$$1 + i + k\left[\frac{k-i}{k}X(M) - 1\right]\frac{H_2'(X(M))}{H_2(X(M))} = (1+i)MX'(M)\frac{H_2''(X(M))}{H_2'(X(M))}.$$

Set $M = e^m$ and define $x(m) = X(e^m)$. Function x is solution of the autonomous ODE

$$(1+i)x'(m)\frac{H_2''(x(m))}{H_2'(x(m))} = \varphi_2(x(m)). \tag{10}$$

with $x(\overline{m}) = 0$, where $\varphi_2(x) = 1 + i + k\left[\frac{k-i}{k}x - 1\right]\frac{H_2'(x)}{H_2(x)}$ and $\overline{m} = \ln \overline{M}$. Recall that function φ_2 is decreasing on $[\frac{k-i}{k}, \infty)$ and φ_2 is positive on $(0, x^*)$ and negative on (x^*, ∞). If at some point $m_0 < \overline{m}$ we have $\varphi_2(x(m_0)) < 0$, then as $\frac{H_2''}{H_2'} < 0$, function x will be increasing on $[m_0, \overline{m}]$ and we shall have $x(m) > x^*$, which leads to a contradiction. Hence, for all $m \in [0, \overline{m}]$, $x'(m) < 0$ and $x(m) < x^*$. Furthermore, integrating ODE (10) and using the fact $x(\overline{m}) = 0$ leads to

$$\int_0^{x(m)} -\frac{H_2''(y)}{H_2'(y)}\frac{dy}{\varphi_2(y)} = \frac{\overline{m} - m}{1+i}, \tag{11}$$

for all $m < \overline{m}$. Relationship (11) fully characterizes function x. It remains to show that as $\overline{m}$ goes to infinity, function x converges to a constant, more specifically x^*. Given $\overline{m}$, function x takes values in the bounded interval $[0, x^*]$. For all $m > 0$, as the right hand side of relationship (11) goes to infinity and $x(m)$ is bounded, the integral on the left hand side must not converge to a finite value. This implies that, for all m, we must have $\lim_{\overline{m}\to\infty} x(m) = x^*$ as $\varphi_2(x^*) = 0$, otherwise the integral on the left hand side of relationship (11) will take a finite value. Once x is known, we can recover function K and verify that $\lim_{\overline{M}\to\infty} K(M) = -\frac{M}{H_2'(x^*)}$. We conclude that as $\overline{M}$ goes to infinity, function $\overline{F}$ converges to function F, our candidate function for the solution of program (P). Finally, observe that $\overline{F}_1 > 0$, $\overline{F}_{11} = -\frac{1}{J_{11}} < 0$. Then, by proceeding in the same way as for function F (see Appendix) one can show that for all $W \leq M$ $F_1(W, M) \leq \overline{K}_0(K(M))^{\frac{1}{1-\beta_2}} W^{\frac{1}{\beta_2 - 1}}$, with $\overline{K}_0 = (-\beta_2 B(-\beta_2, 1+\beta_1))^{\frac{1}{1-\beta_2}}$ and $\overline{\pi}^* \leq \frac{\mu - r}{\sigma^2}(1-\beta_2)$. This implies that $W\overline{\pi}^*$ and $W\overline{\pi}^* F_1$ are bounded on $\left[0, \overline{M}\right]^2$.

Step 2: Let $T > 0$ and denote $\widehat{\tau} = \overline{\tau} \wedge T$. For $t \leq \widehat{\tau}$, applying Itô's lemma for semi-martingales (see for instance, Grossman and Zhou (1993)), we have

$$\begin{aligned} Q_t^\pi &= Q_0^\pi + \int_0^t e^{-(\theta+\delta)s}\mathcal{A}F(W_s^\pi, M_s^\pi)ds + \int_0^t \sigma\pi_s W_s^\pi F_1(W_s^\pi, M_s^\pi)dw_s \\ &+ \int_0^t \left[k-(k-i)F_1(W_s^\pi, M_s^\pi) + (1+i)F_2(W_s^\pi, M_s^\pi)\right] dM_s^\pi, \end{aligned}$$

where

$$\mathcal{A}F(W^\pi, M^\pi) = c_F W^\pi + \frac{\sigma^2}{2}\pi^2 W^2 F_{11} + \pi W^\pi(\mu - r)F_1 - (\theta+\delta)F \leq 0,$$

for all strategy π and equal to 0 for $\pi^* = -\frac{\mu-r}{\sigma^2}\frac{F_1}{WF_{11}}$. It follows that

$$\int_0^{\widehat{\tau}} \sigma\pi_s W_s^\pi F_1(W_s^\pi, M_s^\pi)dw_s \geq Q_{\widehat{\tau}}^\pi - Q_0^\pi \geq -Q_0^\pi \text{ as } Q_{\widehat{\tau}}^\pi \geq 0. \tag{12}$$

The left hand side of the inequality is a local martingale that is bounded from below and hence a supermartigale. Thus

$$0 \geq E_0\left[\int_0^{\widehat{\tau}} \sigma\pi_s W_s^\pi F_1(W_s^\pi, M_s^\pi)dw_s\right] \geq E_0[Q_{\widehat{\tau}}^\pi] - Q_0^\pi,$$

i.e.

$$Q_0^\pi \geq E_0[Q_{\widehat{\tau}}^\pi].$$

For strategy π^*, since $W\overline{\pi}^*$ and $W\overline{\pi}^* F_1^{\pi^*}$ are bounded, the previous inequality is an equality as the left hand side in relationship (12) is actually a martingale. Then, by Lebesgue Monotone Convergence Theorem, we have

$$\lim_{T\to\infty} E_0[Q_{\widehat{\tau}}^{\pi^*}] = Q_0^{\pi^*} \geq Q_0^\pi \geq \lim_{T\to\infty} E_0[Q_{\widehat{\tau}}^\pi].$$

The left hand side of the inequality converges to $E_0\left[\int_0^{\overline{\tau}}(c_F W_s^{\pi^*}ds + kdM_s^{\pi^*})e^{-(\theta+\delta)s}\right] = \overline{F}(W_0, M_0)$, whereas the right hand side of the inequality converges to $E_0\left[\int_0^{\overline{\tau}}(c_F W_s^\pi ds + kdM_s^\pi)e^{-(\theta+\delta)s}\right]$ as for any admissible investment strategy must be such that the corresponding value function satisfies the transversality condition (2). Finally, letting $\overline{M}$ goes to infinity and using once again Lebesgue Monotone Convergence Theorem combined with the fact that $\overline{F}$ converges to F, we obtain that

$$F(W, M) \geq E_0\left[\int_0^{\infty}(c_F W_s^\pi ds + kdM_s^\pi)e^{-(\theta+\delta)s}\right],$$

for every feasible investment strategy π. This concludes the proof. ■

4 Extension to the Baseline Model

We extend our analysis by incorporating an endogenous termination threat of the fund. We assume that the fund's manager cannot experience a large wealth drawdown otherwise clients will withdraw all their wealth. Grossman and Zhou (1993) argue that when "leverage is used extensively, [...] an essential aspect of the evaluation of investment managers and their strategies is the extent to which large drawdowns occur. It is not unusual for the managers to be fired subsequent to achieving a large drawdown (typically above 25 percent)." In this section, we assume that the AUM must satisfy

$$W_t \geq \alpha M_t \text{ for all } t \geq 0, \tag{13}$$

with $\alpha \in [0, 1)$, otherwise the fund is liquidated and the manager receives no severance. Goetzman, Ingersoll and Ross (2003) and Lan, Wang and Yang (2013) use a similar termination condition. We would like to emphasize that condition (13) differs from the one imposed in the two aforementioned papers as at $W_t = \alpha M_t$, liquidation does not take place. In fact, it is never optimal for the fund manager to trigger liquidation, which has interesting implications on the optimal investment strategy.

Define stopping time

$$\tau_L = \inf_{t \geq 0} \{W_t < \alpha M_t\},$$

so that the fund manager' optimization problem now is

$$F(W_0, M_0) = \max_{\pi} E_0 \left[\int_0^{\tau_L \wedge \infty} e^{-(\theta+\delta)t} (c_F W_t dt + k dM_t) \right], \tag{P'}$$

subject to (1).

We expect several implications on the optimal investment strategy. First, as the AUM level is approaching its termination floor, the fund manager's risk aversion should rise, which will curb her position in risky asset in sharp contrast with the baseline model. Second, the manager has now additional reason to mitigate the growth of the HWM due to the ratchet feature of the termination floor.

In order to get some insight, we examine the special case where $c_F = 0$.

4.1 No Management Fee $c_F = 0$

4.1.1 Value Function

For all $u \in [0.1)$, the reduced HJB is:

$$(\theta + \delta) f_\alpha(u) = r u f'_\alpha(u) - \frac{1}{2} \frac{(\mu - r)^2}{\sigma^2} \frac{(f'_\alpha(u))^2}{f''_\alpha(u)}. \tag{14}$$

The general solution of the dual HJB is

$$J_\alpha(x) = K_1 x^{\beta_1} + K_2 x^{\beta_2},$$

so that

$$f_\alpha(u) = (1 - \beta_1) K_1 x^{\beta_1} + (1 - \beta_2) K_2 x^{\beta_2},$$

and

$$u = -\beta_1 K_1 x^{\beta_1 - 1} - \beta_2 K_2 x^{\beta_2 - 1}.$$

Let denote $x_\alpha^* = f_\alpha'(1)$ and $x_\alpha^{**} = f_\alpha'(\alpha)$. At $u = \alpha$, in order not to violate the drawdown constraint with some positive probability in the future, all the AUM must be invested in the riskless asset. This implies that we must have $J_\alpha''(x_\alpha^{**})$ or equivalently $\alpha x_\alpha^{**}(1 - \beta_1 - \beta_2) = \beta_1\beta_2 J_\alpha(x_\alpha^{**})$. Note that this condition is different that the one imposed by Lan, Wang and Yang (2013) - namely $f_\alpha(\alpha) = 0$ - as in our setting, $f_\alpha(\alpha) = J_\alpha(x_\alpha^{**}) + \alpha x_\alpha^{**} \neq 0$. Using the boundary condition at $u = 1$, we obtain the following system (S_0)

$$\begin{aligned}
\alpha &= -\beta_1 K_1 (x_\alpha^{**})^{\beta_1 - 1} - \beta_2 K_2 (x_\alpha^{**})^{\beta_2 - 1} \\
0 &= \beta_1(\beta_1 - 1) K_1 (x_\alpha^{**})^{\beta_1 - 1} + \beta_2(\beta_2 - 1) K_2 (x_\alpha^{**})^{\beta_2 - 1} \\
1 &= -\beta_1 K_1 (x_\alpha^{*})^{\beta_1 - 1} - \beta_2 K_2 (x_\alpha^{*})^{\beta_2 - 1} \\
\frac{k - i}{1 + i} x_\alpha^* &= \frac{k}{1 + i} + K_1 (x_\alpha^*)^{\beta_1} + K_2 (x_\alpha^*)^{\beta_2}.
\end{aligned}$$

In the Appendix, we show that $\varpi = \frac{x_\alpha^*}{x_\alpha^{**}}$ in $(0, 1)$ is implicitly (and uniquely) defined by

$$\alpha[(1 - \beta_2)\varpi^{\beta_1 - 1} + (\beta_1 - 1)\varpi^{\beta_2 - 1}] = \beta_1 - \beta_2.$$

Set $\varpi^* = \left[\frac{\beta_1((k-i)\beta_2 + 1 + i)}{\alpha(1+i)(1-\beta_2)} \vee 0\right]^{\frac{1}{\beta_1 - 1}} \geq 0$. As imposing a drawdown constraint limits the growth of the AUM, assumption A.1. can be weakened as follows:

A1'. Growth condition: $\varpi > \varpi^*$.

Full details on the existence and uniqueness of a solution of system (S_0) are reported in the Appendix. Note that condition A3. is required in this setting to ensure that the wealth process bounces back when W hits the floor αM.

4.1.2 Optimal Investment Strategy

We have the following proposition.

Proposition 7 *The optimal fraction of the AUM π_α^* invested in the risky asset is increasing in u and for all $u \in [0, 1]$ and $\frac{\partial \pi_\alpha^*}{\partial \alpha} < 0$.*

Proof. See the Appendix. ■

Clearly, imposing a threat of termination triggered by a large drawdown of the AUM has a major impact on the optimal investment policy. First, the fear of termination overcomes the fund manager's appetite for risk shifting. The lifetime manager's relative risk aversion becomes decreasing in wealth and consequently the optimal fraction of the AUM invested in risky asset π_α^* is now increasing in u instead of being constant as in Panageas and Westerfield (2009). Second, as shown in Figure 5, the level of risk exposure is also sharply affected: the lifetime manager's relative risk aversion is (globally) magnified by the fear of liquidation and consequently the fraction of AUM invested in the risky asset is all the more (uniformly) reduced as the termination threat becomes more stringent (larger value for α).

4.2 General Case

4.2.1 Value Function

We need to look for a general solution for the reduced dual HJB (6). Details are reported in the Appendix. The general solution is

$$J_\alpha(x) = K_1 H_1(x) + K_2 H_2(x),$$

where K_1 and K_2 are constants to be determined. It follows that

$$u = -J'_\alpha(x) = -K_1 H'_1(x) - K_2 H'_2(x).$$

The boundary conditions at $u = \alpha$ and $u = 1$ (resp. at x^{**}_α and x^*_α) lead to the following system (S)

$$\begin{aligned}
\alpha &= -K_1 H'_1(x^{**}_\alpha) - K_2 H'_2(x^{**}_\alpha) \\
0 &= K_1 H''_1(x^{**}_\alpha) + K_2 H''_2(x^{**}_\alpha) \\
1 &= -K_1 H'_1(x^*_\alpha) - K_2 H'_2(x^*_\alpha) \\
\frac{k-i}{1+i} x^*_\alpha &= \frac{k}{1+i} + K_1 H_1(x^*_\alpha) + K_2 H_2(x^*_\alpha).
\end{aligned}$$

Proposition 8 *For all $\alpha \in (0,1)$, system (S) admits a unique solution with $x^*_\alpha < x^{**}_\alpha$ and the reduced dual value function J_α defined on the interval $[x^*_\alpha, x^{**}_\alpha]$ is decreasing and strictly convex and is given by*

$$J_\alpha(x) = K_1 H_1(x) + K_2 H_2(x),$$

with $K_1 < 0$ and $K_2 > 0$.

Proof. See the Appendix. ■

The optimal (reduced) wealth process is now given by

$$u = -K_1 H'_1(x) - K_2 H'_2(x),$$

where process x is defined in (7). The first term is positive and encapsulates hedging motives to ensure that AUM does fall below the liquidation floor. Typically, this term has a put option flavor and can be related to portfolio insurance strategies involving simple options such as in Black and Perold (1992) and Tepla (2001). As in the baseline model, the second term regulates the growth rate of the AUM to mitigate the ratchet effect of the HWM. In the limiting case $\alpha = 1$, we have

$$x_\alpha^* = x_\alpha^{**} = \frac{\Lambda(1+i) - k\beta_1\beta_2}{(1+i)(1-\beta_1-\beta_2) - (k-i)\beta_1\beta_2} > 0,$$

i.e., $W_t \equiv M_t$ and

$$F(W,M) = \frac{\Lambda(1+i) - k\beta_1\beta_2}{(1+i)(1-\beta_1-\beta_2) - (k-i)\beta_1\beta_2} M.$$

It is easy to verify that in this case F is independent of μ and r.

4.2.2 Optimal Investment Strategy

The optimal investment strategy π_α^* is given by

$$\pi_\alpha^* = -\frac{\mu - r}{\sigma^2} \frac{x J_\alpha''(x)}{J_\alpha'(x)}, \; x_\alpha^* \le x \le x_\alpha^{**}.$$

As in the no management fee case, π_α^* may be increasing in u but alternatively it can be hump-shaped in u; a sufficient condition for the latter to occur is $\left.\frac{\partial \pi_\alpha^*}{\partial x}\right|_{x=x_\alpha^*} > 0$, i.e.

$$J_\alpha''(x_\alpha^*) + x_\alpha^* J_\alpha'''(x_\alpha^*) + x_\alpha^* (J_\alpha''(x_\alpha^*))^2 < 0.$$

Even though we do not report further results, numerical simulations for the baseline case parameters reveal that this condition is always satisfied for sufficiently (very) small values of the drawdown coefficient α. The intuition for such a result is straightforward: The appetite for risk of the hedge fund manager is greatly reduced when the AUM approaches the termination floor but as soon as the AUM is moving away from the floor, the optimal investment strategy exhibits excessive risk taking behavior although at a decreasing rate.

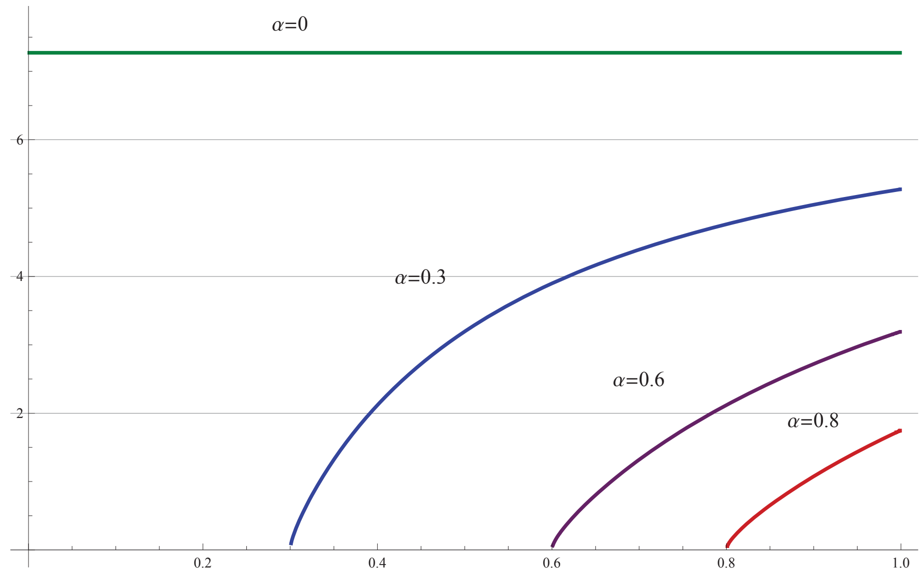

Figure 5 : Fraction of the AUM invested in stocks as a function of $\frac{W}{M}$ $(c_F = 0)$

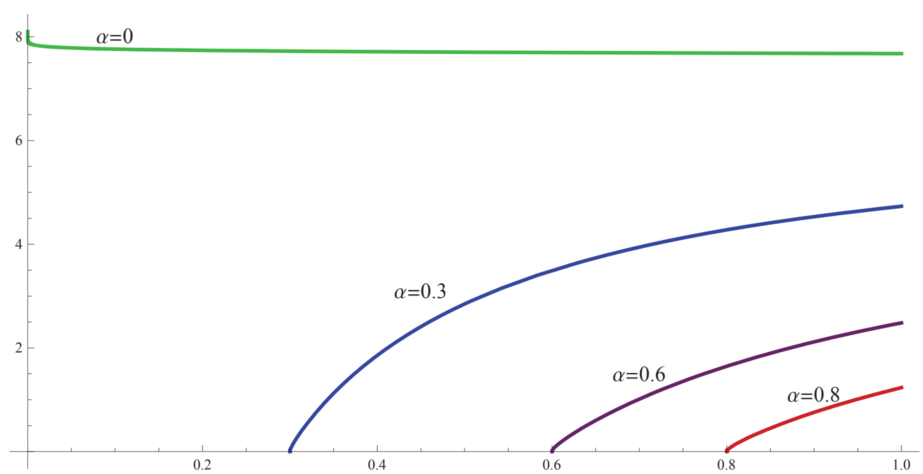

Figure 6 : Fraction of the AUM invested in stocks as a function of $\frac{W}{M}$ $(c_F > 0)$

Comparing figure 5 and figure 6, we note that for sufficiently small values of α, the risk exposure is (globally) larger when a management fee is charged whereas for sufficiently large values of α, the reverse is true.

4.2.3 Fund Manager Compensation Decomposition

As in the baseline case, let f_c and f_k denote the lifetime cumulative management and performance fees respectively and, g_c and g_k the associated functions. The only difference with respect to the baseline case is the boundary condition at $u = \alpha$ or equivalently at $x = x_\alpha^{**}$. Note that $g_c'(x) = -J_\alpha''(x) f_c'(u)$, so it must be the case that $g_c'(x_\alpha^{**}) = 0$ as $J_\alpha''(x_\alpha^{**}) = 0$. Similarly, we must have $g_k'(x_\alpha^{**}) = 0$. In particular, this last boundary condition implies that $f_k(\alpha) = 0$ (but $f_c(\alpha) > 0$). In the Appendix, we show that for all $x \in [x_\alpha^*, x_\alpha^{**}]$, we have

$$g_k(x) = A_{k_1} H_1(x) + A_{k_2} H_2(x),$$

with

$$\begin{aligned}
A_{k_1} &= -\frac{k J_\alpha''(x_\alpha^*) H_2'(x_\alpha^{**})}{(1+k)\left[H_1'(x_\alpha^*) H_2'(x_\alpha^{**}) - H_2'(x_\alpha^*) H_1'(x_\alpha^{**})\right] + (1+i) J_\alpha''(x_\alpha^*) \left[H_1(x_\alpha^*) H_2'(x_\alpha^{**}) - H_2(x_\alpha^*) H_1'(x_\alpha^{**})\right]} \\
&= -\frac{k J_\alpha''(x_\alpha^*)}{(1+k) H_2'(x_\alpha^*) \left[\frac{H_1'(x_\alpha^*)}{H_2'(x_\alpha^*)} - \frac{H_1'(x_\alpha^{**})}{H_2'(x_\alpha^{**})}\right] + (1+i) J_\alpha''(x_\alpha^*) H_2(x_\alpha^*) \left[\frac{H_1(x_\alpha^*)}{H_2(x_\alpha^*)} - \frac{H_1'(x_\alpha^{**})}{H_2'(x_\alpha^{**})}\right]} \\
A_{k_2} &= -\frac{k J_\alpha''(x_\alpha^*)}{(1+k) H_1'(x_\alpha^*) \left[\frac{H_2'(x_\alpha^*)}{H_1'(x_\alpha^*)} - \frac{H_2'(x_\alpha^{**})}{H_1'(x_\alpha^{**})}\right] + (1+i) J_\alpha''(x_\alpha^*) H_1(x_\alpha^*) \left[\frac{H_2(x_\alpha^*)}{H_1(x_\alpha^*)} - \frac{H_2'(x_\alpha^{**})}{H_1'(x_\alpha^{**})}\right]}.
\end{aligned}$$

For the baseline set of parameters, numerical simulations (not reported here) reveal that the discounted value of cumulative incentive fee is quite low and accounts only for a small fraction of the manager total fee compensation.

5 Empirical Evidence

The data were obtained from the Hedge Fund Research database and comprises monthly observations of returns of both active and liquidated hedge funds over the 1976-2013 period. It also includes several fund characteristics, importantly regarding the compensation structure. We drop the observations with non-positive assets and age, and missing data for the basic variables. The sample to funds excludes those that do not have a hurdle rate because of the complexity of computing such hurdle for over six hundred different rates. We look only at funds that do have incentive fees with high-water mark provisions. The final sample consists of 34,919 observations corresponding to 6,267 different funds. All the variables are winsorized at the 1% level.

As stressed by Joenväärä et.al (2016), the relevance of different biases vary across datasets − BarclayHedge, TASS, HFR, EurekaHedge, and Morningstar, in particular − and may alter some conclusions on the performance of funds. We feel, however, that the problem is of less importance when one explores volatility since this is not as salient as returns. For instance, if agents observe or care more about returns than the risks associated to them, funds will not self-select or misreport as much on this variable. Similarly, backfilling and survivorship biases will matter less if they are more related to returns than to volatility.

Risk-taking is not observable and thus has to be estimated. We measure the degree of risk that each fund takes with the volatility of realized monthly returns for the 12 months that follow the anniversary of the inception of the fund. Moreover, since the distance to the HWM is not reported, we follow Aragon and Nanda (2012), and compute it assuming the fund at inception is at the water mark and calculate the HWM in each period as the maximum between the one the year before and the actual value of the fund. The actual value of the fund is the cumulative after-fees return. In order to facilitate the reporting of our results, contrary to the theoretical model, our distance variable is the value of the fund over the HWM minus one, that is, it corresponds to 0 when the fund is at the HWM and decreases as the fund moves farther away from it.

The main simplification here is that we compute only one high-water mark for each fund and period, even though there are many such marks depending on when each investor invested in the fund. This can be an issue for funds with many investment rounds for which the errors-in-variables problem would be a greater issue. However, as these funds are probably older, our age control would ease this concern.

Our benchmark regression model is as follows:

$$\begin{aligned}\text{risk}_{i,t+1} \quad = \quad & \alpha + \beta_1 \times \ln(\text{age})_{i,t} + \beta_2 \times \text{US}_i + \beta_3 \times \text{Return}_{i,t} + \beta_4 \times \text{Rank}_{i,t} + \\ & \beta_5 \times \text{distance to HWM}_{i,t} + \text{year fixed effects } + \text{fund fixed effects } + \varepsilon_{i,t}.\end{aligned}$$

Proposition 4 implies that β_5 is negative since the farther away the fund drifts from the HWM, the smaller the management fee and the present value of the incentive fee will be. If the AUM is far away from the HWM, the infinitely-lived manager will be less worried about surpassing the HWM by too much, which incentivizes risk-taking behavior.

The specification also includes past absolute returns (*Return*) and returns relative to other funds (*Rank*). Empirically, returns relative to other funds have been shown to be of first order importance (see, for instance, Brown etal (2001) and Aragon and Nanda (2012)). Theoretical models suggest their importance, although it is often difficult to separate absolute returns from returns relative to the high-water mark. We will be looking at the effect of the distance to HWM after controlling for these. We also include age to account for potential career concerns and reputational effects and whether the fund is based on the U.S. to accommodate institutional differences in the capacity to take risk and other conventions (when not including fund fixed effects). We also add year fixed effects to capture changing conditions that affect all funds, such as market swings, that may affect performance, and therefore their distance to the watermark and potentially the attitude of portfolio managers towards taking risk.

In our main specification, we use fund fixed effects to capture time-invariant fund characteristics; these include features such as strategy, localization, etc. Controlling for strategy is also important to ease the potential bias arising, for instance, from stale prices due to illiquidity. The identification relies on the within fund variation as we are comparing funds with themselves at different moments in time. In the end, we are asking whether a particular fund behaved differently when at different distances from the high watermark. More specifically, if the fund took more risk in the years in which it was farther away from its HWM.

Previous empirical work has adopted a framework that is a bit different from ours in that is has sought to explain changes in risk-taking from the first to the second semester with the distance to the HWM at mid year (see, for instance, Brown et.al (2001), Agarwal et.al (2002), and Aragon and Nanda (2012)). We believe our framework is more in the spirit of our model where the manager is infinitely lived and is not concerned just with the short-term: she will have an incentive to take on more risk whenever she is under the HWM and not just (or especially) during the second semester of each year. Also, we are not forced to assume that HWM are always resent on January.

Also, when looking at the impact of fees, many papers look at differences in risk-taking across funds with and without incentive fees. Here we focus on funds with incentive fees and HWM provisions and explore whether the level of fees make a difference on risk-taking.

The downside of our specification is that we will not be able to identify the effect of time-invariant (or nearly) fund characteristics; importantly the impact of the structure of fees. We will, however, present results with no fund fixed effects to provide suggestive evidence of the likely impact of those factors.

As a robustness check, we show that previous specifications provide results that are qualitative and (for the most part) quantitatively consistent. We compute robust standard errors to consider potential heteroskedasticity and cluster them at the management firm level[12].

Table 1. Summary Statistics

	Obs	Mean	S.D.	Min	Max
ST Dev of Return ex-post	34,919	0.117	0.097	0.009	0.497
Return t-1	34,919	0.128	0.211	-0.428	0.885
Rank t-1	34,919	1,861	1,296	1	4,885
Assets (milion USD)	33,906	190	590	0.01	26,326
Age	34,919	5.2	4.2	1.0	35.0
U.S. based	34,919	0.76	0.43	0.00	1.00
Distance to HWM	34,919	-0.027	0.082	-0.507	0.000
Under HWM	34,919	0.19	0.39	0.00	1.00
Mgmt Fee %	34,919	1.47	0.59	0.00	10.00
Incentive Fee %	34,919	18.3	4.7	1.0	50.0
Year	34,919	2,005	5	1,978	2,012

Table 1 summarizes the data. There is important variation around the mean ex-post volatility of returns of 11.7% that we can exploit: the standard deviation is 9.7%, the minimum 0.9% and the maximum 49.7%. The average fund size is 190 million dollars and 5.2 years old. Seventy-six percent of them are based in the U.S. As for the fees, these are not far from the traditional 2/20 structure, with a median of 1.5% for the management fee and 20% for the incentive remuneration. 6,635 observations

[12] Aragon and Nanda (2012) also cluster by strategy. Results are mainly unaffected, but the incentive fee effect turns out to be non-significant.

(19% of the sample) correspond to years in which a fund was under its HWM. Overall, these figures are similar to the ones reported in the literature using other datasets and time frames. When one considers all the funds, the average distance to the HWM is 2.7%. Conditional on being underwater, the mean distance is 14.3%, with a standard deviation of 13.8%.

On average, when a fund surpasses its HWM, it does so by a margin of 12.7% (standard deviation 16.9%) and once it does, it takes on average 1.22 years to reach it again. Having the fee structure of a fund, one can compute the share of income that comes from the incentive and the management component during a given period. If we consider the period between two hits, a fund manager will get on average 48.5% of its income from the incentive component. That is, incentive fees are a significant fraction of revenues and, therefore, should be expected to play an important role in the strategic behavior of fund managers, as our model suggests.

Table 2. Benchmark Regressions

The dependent variable is the standard deviation of monthly returns during the 12 months that follow the aniversary of each fund's inception date. Distance to HWM is the value of the fund divided by the HWM minus one, that is, it corresponds to 0 when the fund is at the HWM and decreases as the fund moves farther away from it. All variables are winsorized at the 1% level. Standard errors are robust to heteroskedasticity and clustered at the management firm level. Significance levels: * p<0.05, ** p<0.01, *** p<0.001

Log age (years)	0.001	0.004***	0.003	0.004***	0.004***	0.003
	-0.001	-0.001	-0.002	-0.001	-0.001	-0.002
U.S. based	(-0.005)	(-0.004)		(-0.003)	(-0.003)	
	-0.004	-0.003		-0.004	-0.004	
Distance to HWM	-0.283***	-0.468***	-0.096***	-0.465***	-0.249***	0.032
	-0.012	-0.015	-0.01	-0.015	-0.043	-0.034
Return t-1		0.173***	0.045***	0.172***	0.172***	0.045***
		-0.006	-0.005	-0.006	-0.006	-0.005
Rank t-1 x 10-3		-0.005***	-0.006***	-0.006***	-0.006***	-0.006***
		-0.001	-0.001	-0.001	-0.001	-0.001
Management fee				0.011***	0.010***	
				-0.004	-0.003	
Incentive fee				0.001*	0.001	
				-0.001	-0.001	
Distance to HWM x Management fee					-0.018	-0.002
					-0.02	-0.021
Distance to HWM x Incentive fee					-0.010***	-0.007***
					-0.002	-0.002
N	34919	34919	34919	34919	34919	34919
R-sqr	0.262	0.336	0.724	0.341	0.343	0.724
Strategy fixed effects	YES	YES	YES	YES	YES	YES
Fund fixed effects	NO	NO	YES	NO	NO	YES
Year fixed effects	YES	YES	YES	YES	YES	YES

Table 2 presents the main results. To explore the main effects of time-invariant fund characteristics, importantly the fee structure, we do not include fund fixed effects in some of the regressions in the table. We then corroborate that the results are not due to time-invariant omitted variable bias by adding the fund fixed effects.

The first column establishes that risk-taking increases with the distance to the HWM: the coefficient of this variable is negative and highly significant. Consistent with our model, funds that are further away from their HWM tend to take more risk when compared to others that are closer to it. Being older and based in the U.S. does not seem to have a major impact on risk. The second column shows that the positive relation between distance and risk is not (entirely) driven by the absolute return of the fund or its return in relation to the others: the coefficient for distance is still negative and significant. The negative sign for the *Rank* variable is consistent with tournament behavior that has been documented before: funds that do poorly relative to others tend to take on more risk. The positive effect of *Return* means that funds that do well take tend to be riskier. This is not inconsistent with previous findings such as Aragon and Nanda (2012)'s since this effect is after controlling for performance relative to others and relative to oneself. Consistent with what Brown et.al (2001) finds, the significance of the negative relation between relative performance and volatility, although it does not disappear completely, drops to half when one controls for the distance to HWM (not reported).

The results above come from pooled regressions, that is, are identified via comparing the same fund at different moments but also across different funds. This can be problematic since funds do not only vary in terms of their distance to HWM and leaving those other characteristics aside may induce estimation bias. In the third column we add fund fixed effects and show that the relation between distance to HWM and risk is still consistent with the prediction of the model when we identify the effect just by comparing the same fund at different moments: the coefficient for the value relative to HWM is still strongly negative. Importantly, the effects of distance and of returns are independent of each other. That is, we are not simply capturing the fact that risk increases after good performance. Rather, we document that the effect is reinforced when the fund is farther away from its high-water mark. Its magnitude is about one third of the one obtained before, meaning that other, time-invariant fund characteristics were explaining the bulk of the effect, highlighting the importance of controlling for these in empirical work. Despite this, the economic magnitude of the effect is important. For instance, being 20% underwater is associated with 192 bps increase in the standard deviation of the next 12-month returns or 16.4%.

In the following columns we explore the role of the fee structure on risk-taking. Column four reports that risk increases with the level of management fee since the coefficient is positive and significant. This is what we expected from the model and the simulations ever since the management fee acts as an insurance. Increasing the management fee rate from, say, 1% to 3% is associated with an increase in 20% of the standard deviation of returns. The positive coefficient for the incentive fee rate is, however, not consistent with the model. We expected a negative coefficient because if the manager takes on more risk and beats the HWM by a larger amount, it becomes harder to beat it again in the future. In any case, the coefficient is only marginally significant (p-value 9%) and very small: increasing the fee from 15% to 20% would increase risk by only 42 bps or 5%. Moreover, as can be seen in the next column, the result is not robust either.

In column 5 we explore how the relation between the distance to the HWM and risk is shaped by the structure of fees. Our simulation depicted in Figure 2 suggests that the increase in risk as the fund gets farther away from the HWM is all the more severe as the incentive fee gets larger as moving farther away from the HWM implies a larger forgone present value of the incentive fee.

We therefore expect the coefficient of *Distance* to be more negative for funds that charge a higher incentive fee. This is exactly what the negative and significant coefficient for the interaction between *Distance* and *Incentive Fee* means. Its size is relevant: the slope of risk to distance is 20% higher for a fund with a 20% incentive fee compared to one with a 15% charge. This expands Aragon and Nanda (2012)'s results as the relationship not only gets stronger for funds with incentive pay but increase with the level of it. On the contrary, we do not find the effect of distance being stronger with the level of management fee.

From now on we include fund-fixed effects in all regressions and show that the results are robust to controlling for all time-invariant fund characteristics. Of course, we are no longer able to identify the main effect of the fee structure. Column six shows that these effects are robust to controlling for fund fixed effects. The magnitude of the incentive fee distance interaction is of similar magnitude.

Table 3. Further Results

Panel A

The dependent variable is the standard deviation of monthly returns during the 12 months that follow the aniversary of each fund's inception date. Distance to HWM is the value of the fund divided by the HWM minus one, that is, it corresponds to 0 when the fund is at the HWM and decreases as the fund moves farther away from it. All variables are winsorized at the 1% level. Standard errors are robust to heteroskedasticity and clustered at the management firm level.

Significance levels: * p<0.05, ** p<0.01, *** p<0.001

		Equity Hedge	Event-Driven	Fund of Funds	Macro	Relative Value
Log age (years)	0.003	0.001	0.001	0.004*	-0.001	0.001
	(0.002)	(0.002)	(0.004)	(0.002)	(0.005)	(0.004)
Distance to HWM	-0.049**	-0.093***	-0.149***	-0.012	-0.126***	-0.052*
	(0.023)	(0.013)	(0.035)	(0.025)	(0.032)	(0.029)
Distance to HWM squared	0.117**					
	(0.053)					
Return t-1	0.044***	0.039***	0.047***	0.042***	0.035***	0.030**
	(0.005)	(0.007)	(0.015)	(0.015)	(0.012)	(0.012)
Rank t-1 x 10^{-3}	-0.006***	-0.004***	-0.008***	-0.003***	-0.004***	-0.009***
	(0.001)	(0.001)	(0.002)	(0.001)	(0.002)	(0.001)
N	34919	14950	3941	4974	6593	4461
R-sqr	0.724	0.704	0.7	0.76	0.718	0.674
Strategy fixed effects	YES	YES	YES	YES	YES	YES
Fund fixed effects	YES	YES	YES	YES	YES	YES
Year fixed effects	YES	YES	YES	YES	YES	YES

In Table 3 we expand the results. First, we find that the distance effect is not linear but also convex: as the fund gets farther away from the HWM, the incentive risk-taking increases more rapidly. In the first column of Panel A, we show that the square of distance enters significantly positive in the regression. Figure 7 depicts this result. This is what we obtain from our simulations.

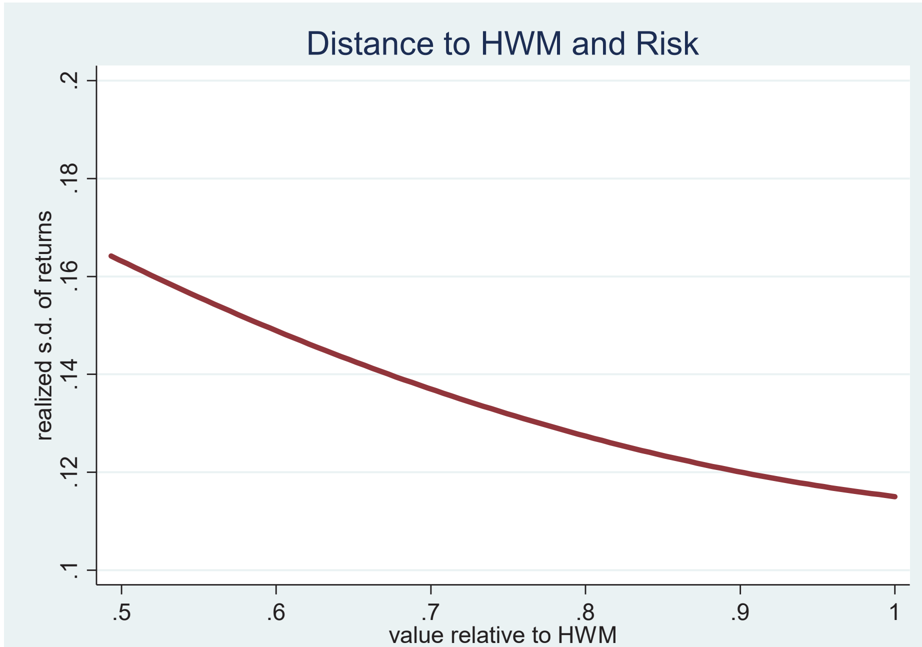

Figure 7 : AUM returns volatility as a function of the distance to HWM

The next columns of Panel A show that, although to different degrees, the main result generally applies to all kinds of hedge funds. Except for the case of funds of funds, the effect of distance for the average fund is negative. Furthermore, the increase in risk following poor performance relative to the HWM is more pronounced for funds with higher incentive fees in most kinds, although not always significantly so (not reported). One would expect the response of managers to be larger when there is more flexibility for them to change the level of risk, in particular, to increase it. We do find such an effect since the impact is stronger for funds can lever up, as reported in the first column in panel B, since the interaction between an indicator of this capacity and distance is significantly negative.

Panel B

The dependent variable is the standard deviation of monthly returns during the 12 months that follow the aniversary of each fund's inception date. Distance to HWM is the value of the fund divided by the HWM minus one, that is, it corresponds to 0 when the fund is at the HWM and decreases as the fund moves farther away from it. Column 1 includes only the funds that are allowed to lever up. All variables are winsorized at the 1% level. Standard errors are robust to heteroskedasticity and clustered at the management firm level.

Significance levels: * p<0.05, ** p<0.01, *** p<0.001

Log age (years)	0.003	0.003	0.003
	(0.002)	(0.002)	(0.002)
	(0.015)	(0.013)	(0.101)
Return t-1	0.045***	0.045***	0.126***
	(0.005)	(0.005)	(0.023)
Rank t-1 x 10^{-3}	-0.006***	-0.006***	-0.059***
	(0.001)	(0.001)	(0.015)
Distance to HWM x Leveraged	-0.028*		
	(0.016)		
Distance to HWM x Fast Redemption		0.037**	
		(0.017)	
Threat of Liquidation			-1.876***
			(0.528)
Distance to HWM x Threat of Liquidation			0.600*
			(0.312)
N	34760	34919	34919
R-sqr	0.724	0.724	0.724
Strategy fixed effects	YES	YES	YES
Fund fixed effects	YES	YES	YES

The threat of a large drawdown following poor performance has a major impact on the optimal policy. In particular, the fear of liquidation magnifies the relative risk aversion of the manager and consequently the fraction invested in the risky asset is reduced. To test this implication, we ask whether the impact of distance on risk is reduced when this is more likely. In column two add the interaction between distance and an indicator that takes the value of 1 if the funds can be redeemed within less than a month and 0 otherwise. The coefficient for that variable is significantly positive, as expected. We did the same exercise for funds needing less than 30-day notice in advance to withdraw the money and those with minimum investment of 1 million dollars and also found positive coefficients, although not significantly so (not reported).

To further test for the impact of the likelihood of liquidation we took a two-step approach. First, we estimated a probit model to predict whether a fund would be liquidated at any point in time. The model has an indicator variable that takes a value of 1 if the fund was actually liquidated and cero otherwise, and two independent variables: absolute return and return relative to the other funds. In the second stage we add the predicted value for liquidation from the first step to our benchmark regression, both alone and interacted with the distance variable. We get a negative coefficient for the threat of liquidation and a positive coefficient for its interaction with distance. This is exactly what we expected from the model.

Table 4 explores what happens with the frequency and extent to which the high-water mark is surpassed.

Table 4. Frequency and Extent of Surpass of HWM

The dependent variable is the time between two hits in columns 1 and 2, and the percentage increase in the HWM in columns 3 and 4. Distance to HWM is the value of the fund divided by the HWM minus one, that is, it corresponds to 0 when the fund is at the HWM and decreases as the fund moves farther away from it. All variables are winsorized at the 1% level. Standard errors are robust to heteroskedasticity and clustered at the management firm level. Significance levels: * p<0.05, ** p<0.01, *** p<0.001

	Time between Hits		Extent of Surpass	
Log age (years)	0.060***	0.060***	-0.035***	-0.035***
	(0.007)	(0.007)	(0.002)	(0.002)
Distance to HWM	-2.452***	-3.028***	-0.461***	-0.658***
	(0.078)	(0.292)	(0.018)	(0.068)
Return t-1	-0.275***	-0.276***	0.600***	0.600***
	(0.023)	(0.023)	(0.013)	(0.013)
Rank t-1 x 10^{-3}	-0.107***	-0.107***	0.007***	0.007***
	(0.005)	(0.005)	(0.001)	(0.001)
Distance to HWM x Management fee		0.047		0.019
		(0.132)		(0.016)
Distance to HWM x Incentive fee		0.027*		0.009**
		(0.014)		(0.004)
N	34045	34045	34819	34819
R-sqr	0.551	0.552	0.688	0.688
Strategy fixed effects	YES	YES	YES	YES
Fund fixed effects	YES	YES	YES	YES
Year fixed effects	YES	YES	YES	YES

The first column shows that the time elapsed between hits increases with the distance to HWM, as reflected in the negative coefficient for the distance to HWM variable. That is, maintaining the AUM is close to its HWM will typically result in hitting the watermark often. This is what was expected from the model: as the fund falls behind the HWM it becomes increasingly difficult to surpass it in the future; taking on more risk mitigates this effect. Also consistent, is the fact that this effect is weaker as the incentive fee increases (column two). We did not find, however, that the frequency of hits decreases more rapidly in funds with higher management fee rates.

The next two columns explore the extent to which the watermark is surpassed when it effectively is. Since the coefficient for distance is negative, the jump is smaller when the fund is closer to its HWM. This is consistent with the intuition because, in that case, it will be optimal to beat the high-water mark frequently by a small amount to mitigate the ratchet effect. Those considerations become less important as the fund gets farther away, and especially when the management fee rate is larger (insurance) and the performance fee rate is smaller (future cost of surpassing the HWM). We find evidence consistent with the latter implication.

In Table A1 in the Appendix, we conduct our benchmark analysis using an alternative specification, in the line of Brown et.al (2001), and Aragon and Nanda (2012). This consists on observing the change in risk during the second semester of the year with respect to the first semester and relating it to the fund's performance. To be consistent with our previous assumptions, in the first thee columns we consider only the funds with inception in the month of January. This greatly reduces the number of observations. The results are perfectly consistent with what has been documented in the literature before: on average risk increases following poor performance measured in relation to others and to the HWM. In the following three, we expand the sample to include all funds, regardless of their inception month. That is, we compute the change in the standard deviation of monthly returns for months $t+6$ through $t+12$ versus months t though $t+5$ relative to each fund's inception date. The results are qualitatively the same compared to our benchmark in Table 2: risk increases with the management fee and with the distance to HWM, especially when the incentive fee is high.

As a robustness check, in column 7 we just keep the funds that have neither and incentive fee nor a HWM provision and show that, for them, there is no effect on risk-taking of being far from what would have been their high-water mark.

6 Conclusion

We have examined how a management fee combined with a performance fee affect the optimal investment strategy of a hedge fund and the hedge fund manager's compensation. Our baseline model is a simple extension of the work by Panageas and Westerfield (2009). One of our main finding is to highlight that the important role played by management fee as it contributes to smooth out the manager's revenues, acting as an insurance policy. Ceteris paribus, this translates into a more aggressive optimal investment strategy with respect to the no management fee case. Second, even though the fund manager has risk neutral preference over money, the option like compensation scheme makes her lifetime utility of the manager exhibits increasing relative risk aversion (IRRA). Consequently, the fraction of the AUM invested in equity is all the more rising the farther away the AUM moves from the HWM. This is in sharp contrast with the result in Panageas and Westerfield (2009) in which the optimal investment strategy consists in holding a constant fraction of the AUM in stock that is independent of the performance fee rate. In this paper, holdings in risky assets are negatively related to the magnitude of the performance fee rate. This reflects the intertemporal trade-off faced by the manager: Rising the AUM volatility by tilting portfolio holdings towards equity in order to significantly, although infrequently, surpass the HWM and pocket a hefty performance fee or alternatively beating more often the HWM by a small amount by choosing a more conservative investment strategy. The latter turns out to be optimal, reflecting the manager's preference for a policy of small steps to achieve smoother revenues. Consistently, we find that the expected time until the next HWM hit is increasing in the (relative) distance between the current value of the AUM and the HWM. Regarding the manager compensation, perhaps surprisingly, we find that both an increase in either the management fee or the performance fee lowers the manager's lifetime earnings but recall that proportional fees reduce the size of the AUM. Regarding the compensation decomposition, we find that the fund manager derives most of her revenue from the performance fee.

An extension to the baseline model introduces an early termination should the AUM experience a sufficiently large drawdown, measured as fraction of the HWM. The impact on the optimal investment strategy is significant. The closer the AUM gets to the minimum floor, the higher the fund manager's lifetime risk aversion, which curbs down the risky portfolio allocation. Depending of the parameters of the model, the optimal fraction of AUM invested in risky asset is either increasing in wealth or hump shaped. The former pattern always prevails when the management fee is small and the liquidation floor is high. Conversely, we observe the latter pattern for sufficiently large management fee rate and low liquidation floor, which indicates that as soon as the termination threat is low enough as the AUM has moved away from the liquidation floor, the convex like feature of the compensation scheme induces the optimal investment strategy to exhibit excess risk taking as in the baseline model.

We provide empirical support for the main implications of the model. Data seem to support the theoretical predictions of the model: returns' volatility is strongly related to distance to the HWM, especially for funds with a high incentive fee rate. Also, the time elapsed between hits and the extent to which the fund surpasses the HWM both increase with distance. Finally, the threat of fund termination reduces risk and mitigates the positive relationship between risk and distance to the high-water mark.

7 Appendix

Appendix A

$F(W, M) < \infty$. For all $0 \leq W_0 \leq M_0$ we have

$$\begin{aligned} F(W_0, M_0) &\leq E_0 \left[\int_0^\infty e^{-(\theta+\delta)t} (c_F M_t dt + k dM_t) \right] \\ &\leq \frac{c_F}{\theta+\delta} M_0 + \frac{(\theta+\delta)c_F + k}{k} \max_\pi E_0 \left[\int_0^\infty e^{-(\theta+\delta)t} k dM_t \right]. \end{aligned}$$

Under condition A1, the expectation on the RHS of the inequality is indeed bounded (see Panageas Westerfield (2009)).

Derivation of the Dual Reduced Value Function J. Consider the following ODE

$$x^2 f''(x) + [(1-\beta_1-\beta_2)x - \Lambda)] f'(x) + \beta_1\beta_2 f(x) = 0, \tag{15}$$

with $\Lambda = -\frac{c_F \beta_1 \beta_2}{\theta+\delta} > 0$. Define auxiliary function g such that $f(x) = x^{\beta_2} g(x)$. It is easy to verify that g satisfies the following ODE

$$x^2 g''(x) + [(1-\beta_1+\beta_2)x - \Lambda)] g'(x) - \beta_2 \frac{\Lambda}{x} g(x) = 0.$$

Then, define auxiliary function h such that $g(x) = h(-\frac{\Lambda}{x})$ and $y = -\frac{\Lambda}{x} < 0$. It is easy to verify that function h is a solution of the Kummer equation

$$yh''(y) + (b-y)h'(y) - ah(y) = 0, \tag{16}$$

where $a = -\beta_2$ and $b = 1 + \beta_1 - \beta_2$. Next we show that if g is a solution on the positive real line of (16) with parameters (a, b), then h defined by $h(y) = e^y g(-y)$ with $y < 0$ is a solution on the negative real line of (16) with parameters $(b-a, b)$. Set $x = -y$, we have $g'(x) = e^x [h(-x) - h'(-x)]$ and $g''(x) = e^x [h(-x) - 2h'(-x) + h''(-x)]$. It is easy to check that h satisfies:

$$yh''(y) + (b-y)h'(y) - (b-a)h(y) = 0.$$

One solution of equation (16) is the Kummer function

$$h(z) = M(a, b, z) = \frac{1}{B(a, b-a)} \int_0^1 e^{zt} t^{a-1} (1-t)^{b-a-1} dt,$$

where $\Gamma(a) = \int_0^\infty e^{-u} u^{a-1} du$ and $B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$ are the Euler Gamma and Beta functions, respectively. An independent solution for $z < 0$ is

$$h(z) = e^z U(b-a, b, -z) = \frac{1}{\Gamma(a)} \int_0^\infty e^{z(1+t)} t^{b-a-1} (1+t)^{a-1} dt.$$

The general solution of (15) is given by:

$$f(x) = K_1 H_1(x) + K_2 H_2(x),$$

where

$$\begin{aligned} H_1(x) &= x^{\beta_2} \int_0^\infty e^{-\frac{\Lambda(1+t)}{x}} t^{\beta_1} (1+t)^{-\beta_2-1} dt \\ H_2(x) &= x^{\beta_2} \int_0^1 e^{-\frac{\Lambda t}{x}} t^{-\beta_2-1} (1-t)^{\beta_1} dt, \end{aligned}$$

with $(K_1, K_2) \in \mathbb{R}^2$. One can check that

$$\begin{aligned} H_1'(x) &= \beta_1 x^{\beta_2-1} \int_0^\infty e^{-\frac{\Lambda(1+t)}{x}} t^{\beta_1-1} (1+t)^{-\beta_2} dt > 0 \\ H_1''(x) &= \beta_1(\beta_1-1) x^{\beta_2-2} \int_0^\infty e^{-\frac{\Lambda(1+t)}{x}} t^{\beta_1-2} (1+t)^{-\beta_2+1} dt > 0 \\ H_2'(x) &= -\beta_1 x^{\beta_2-1} \int_0^1 e^{-\frac{\Lambda t}{x}} t^{-\beta_2} (1-t)^{\beta_1-1} dt < 0 \\ H_2''(x) &= \beta_1(\beta_1-1) x^{\beta_2-2} \int_0^1 e^{-\frac{\Lambda t}{x}} t^{1-\beta_2} (1-t)^{\beta_1-2} dt > 0. \end{aligned}$$

In the paper, we make use of the following asymptotic behaviors

$$\begin{aligned} &H_1(x) \underset{0}{\sim} \Lambda^{-\beta_1-1} \Gamma(\beta_1+1) x^{\beta_1+\beta_2+1} e^{-\frac{\Lambda}{x}} \text{ and } H_1'(x) \underset{0}{\sim} \Lambda^{-\beta_1} \Gamma(\beta_1+1) x^{\beta_1+\beta_2-1} e^{-\frac{\Lambda}{x}} \\ &H_1(x) \underset{\infty}{\sim} \Lambda^{\beta_2-\beta_1} \Gamma(\beta_1-\beta_2) x^{\beta_1} \text{ and } H_1'(x) \underset{\infty}{\sim} \beta_1 \Lambda^{\beta_2-\beta_1} \Gamma(\beta_1-\beta_2) x^{\beta_1-1} \\ &H_2(x) \underset{0}{\sim} \Lambda^{\beta_2} \Gamma(-\beta_2) \text{ and } H_2'(x) \underset{0}{\sim} -\beta_1 \Lambda^{\beta_2-1} \Gamma(1-\beta_2) \\ &H_2(x) \underset{\infty}{\sim} B(-\beta_2, 1+\beta_1) x^{\beta_2} \text{ and } H_2'(x) \underset{\infty}{\sim} \beta_2 B(-\beta_2, 1+\beta_1) x^{\beta_2-1}, \end{aligned}$$

so that we obtain that

$$f(u) \underset{0}{\sim} \frac{\beta_2-1}{\beta_2} Q u^{\frac{\beta_2}{\beta_2-1}}, \text{ with } Q = \left(\frac{\beta_2 B(-\beta_2, 1+\beta_1)}{H_2'(x^*)} \right)^{\frac{1}{1-\beta_2}}. \tag{17}$$

The Wronskian of ODE (15) is given by

$$\begin{aligned} W(H_1,H_2)(x) &= H_2'(x)H_1(x) - H_1'(x)H_2(x) \\ &= -\Lambda^{\beta_2-\beta_1}\Gamma(-\beta_2)\Gamma(\beta_1+1)e^{-\frac{\Lambda}{x}}x^{\beta_1+\beta_2-1} < 0 \\ W(H_1',H_2')(x) &= H_2''(x)H_1'(x) - H_1''(x)H_2'(x) \\ &= \beta_1\beta_2 x^{-2}W(H_1,H_2)(x) > 0. \end{aligned}$$

Assume that the boundary condition at $u = 0$ is reached a finite point $\overline{x} \in (0,\infty)$. We have

$$\begin{aligned} 0 &= K_1H_1(\overline{x}) + K_2H_2(\overline{x}) \\ 0 &= K_1H_1'(\overline{x}) + K_2H_2'(\overline{x}). \end{aligned}$$

The first condition implies that constants K_1 and K_2 must have opposite sign; using the second condition, since $H_1' > 0$ and $H_2' < 0$, we deduce that constants K_1 and K_2 must have the same sign, which leads to a contradiction as both K_1 and K_2 are not equal to zero. We conclude that $\overline{x} \in \{0,\infty\}$. Assume that $\overline{x} = 0$; then, we must have $K_2 = 0$. Since $J' \leq 0$, we must have $K_1 < 0$, which leads to a contradiction as $J \geq 0$. We conclude that $\overline{x} = \infty$, so $K_1 = 0$ and we must have $K_2 > 0$ in order to have $u \geq 0$. The boundary condition at $u = 1$ translates into:

$$\begin{aligned} 1 &= -K_2H_2'(x^*) \\ (k-i)x^*H_2'(x^*) &= kH_2'(x^*) - (1+i)H_2(x^*). \end{aligned}$$

We deduce that interval I must be of the form $[x^*,\infty)$, with $x^* > 1$. Next we establish the existence and uniqueness of x^*. ■

Appendix B

Existence and Uniqueness of x^*. We want to show that function φ_2 has a unique root $x^* > 1$, where

$$\varphi_2(x) = k(ax-1)\frac{H_2'(x)}{H_2(x)} + 1 + i, \tag{18}$$

with $a = \frac{k-i}{k} \in (0,1)$. Let $z = a - \frac{1}{x}$, so that $x = \frac{1}{a-z}$; Define $\phi_2(z) = z\frac{\int_0^1 e^{\Lambda(z-a)t}t^{-\beta_2}(1-t)^{\beta_1-1}dt}{\int_0^1 e^{\Lambda(z-a)t}t^{-\beta_2-1}(1-t)^{\beta_1}dt} = -z + z\frac{\int_0^1 e^{\Lambda(z-a)t}t^{-\beta_2-1}(1-t)^{\beta_1-1}dt}{\int_0^1 e^{\Lambda(z-a)t}t^{-\beta_2-1}(1-t)^{\beta_1}dt}$ and observe that $\varphi_2(x) = k\beta_2\phi_2(z) + 1 + i$. We want to show that ϕ_2 is increasing in z. We have

$$\phi_2'(z) = -1 + \frac{\int_0^1 e^{\Lambda(z-a)t}t^{-\beta_2-1}(1-t)^{\beta_1-1}dt}{\int_0^1 e^{\Lambda(z-a)t}t^{-\beta_2-1}(1-t)^{\beta_1}dt} + \frac{\Lambda z\Phi_2(z)}{D^2},$$

where $D = \int_0^1 e^{\Lambda(z-a)t} t^{-\beta_2 - 1}(1-t)^{\beta_1} dt$ and

$$\begin{aligned}\Phi_2(z) &= \int_0^1 e^{\Lambda(z-a)t} t^{-\beta_2}(1-t)^{\beta_1 - 1} dt \times \int_0^1 e^{\Lambda(z-a)t} t^{-\beta_2 - 1}(1-t)^{\beta_1} dt \\ &\quad - \int_0^1 e^{\Lambda(z-a)t} t^{-\beta_2 - 1}(1-t)^{\beta_1 - 1} dt \times \int_0^1 e^{\Lambda(z-a)t} t^{-\beta_2}(1-t)^{\beta_1} dt.\end{aligned}$$

Then, note that

$$\begin{aligned}\int_0^1 e^{\Lambda(z-a)t} t^{-\beta_2 - 1}(1-t)^{\beta_1} dt &= \int_0^1 e^{\Lambda(z-a)t} t^{-\beta_2 - 1}(1-t)^{\beta_1 - 1} dt - \int_0^1 e^{\Lambda(z-a)t} t^{-\beta_2}(1-t)^{\beta_1 - 1} dt \\ \int_0^1 e^{\Lambda(z-a)t} t^{-\beta_2}(1-t)^{\beta_1} dt &= \int_0^1 e^{\Lambda(z-a)t} t^{-\beta_2}(1-t)^{\beta_1 - 1} dt - \int_0^1 e^{\Lambda(z-a)t} t^{-\beta_2 + 1}(1-t)^{\beta_1 - 1} dt.\end{aligned}$$

It follows that

$$\begin{aligned}\Phi_2(z) &= -\left(\int_0^1 e^{\Lambda(z-a)t} t^{-\beta_2}(1-t)^{\beta_1 - 1} dt\right)^2 \\ &+ \left(\int_0^1 e^{\Lambda(z-a)t} t^{-\beta_2 - 1}(1-t)^{\beta_1 - 1} dt\right) \times \left(\int_0^1 e^{\Lambda(z-a)t} t^{-\beta_2 + 1}(1-t)^{\beta_1 - 1} dt\right) > 0,\end{aligned}$$

by the Cauchy Schwartz inequality with

$$\begin{aligned}f_a(t,z) &= e^{\frac{\Lambda}{2}(z-a)t} t^{-\frac{\beta_2 + 1}{2}}(1-t)^{\frac{\beta_1 - 1}{2}} \\ g_a(t,z) &= e^{\frac{\Lambda}{2}(z-a)t} t^{-\frac{\beta_2 - 1}{2}}(1-t)^{\frac{\beta_1 - 1}{2}},\end{aligned}$$

so that $f_a(t,z) g_a(t,z) = e^{\Lambda(z-a)t} t^{-\beta_2}(1-t)^{\beta_1 - 1}$. Finally, we note that $-1 + \frac{\int_0^1 e^{\Lambda(z-a)t} t^{-\beta_2 - 1}(1-t)^{\beta_1 - 1} dt}{\int_0^1 e^{\Lambda(z-a)t} t^{-\beta_2 - 1}(1-t)^{\beta_1} dt} > 0$. We conclude that $\phi_2' > 0$. It follows easily that $\varphi_2' < 0$. As φ_2 is continuous with $\varphi_2(\frac{k}{k-i}) = 1 + i > 0$ and $\lim_{x \to \infty} \varphi_2(x) = (k-i)\beta_2 + 1 + i < 0$ by assumption A1, we conclude that φ_2 has a unique root $x^* > \frac{k}{k-i} > 1$ and note that $(k-i)\beta_2 + 1 + i < 0$ is necessary and sufficient. ■

$\frac{\partial x^*}{\partial k} < 0$ and $\frac{\partial x^*}{\partial i} > 0$. Totally differentiating relationship (18) with respect to k and evaluating at $x = x^*$ leads to

$$(x^* - 1)\frac{H_2'(x^*)}{H_2(x^*)} + \varphi_2'(x^*)\frac{\partial x^*}{\partial k} = 0.$$

Since $\varphi_2'(x^*) < 0$ and $(x^* - 1)\frac{H_2'(x^*)}{H_2(x^*)} < 0$, we deduce that $\frac{\partial x^*}{\partial k} < 0$. Similarly, totally differentiating $\varphi_2(x^*)$ with respect to i leads to

$$-\frac{x^* H_2'(x^*)}{H_2(x^*)} + 1 + \varphi_2'(x^*)\frac{\partial x^*}{\partial i} = 0.$$

Since $\varphi_2'(x^*) < 0$ and $-\frac{x^* H_2'(x^*)}{H_2(x^*)} + 1 > 0$, we deduce that $\frac{\partial x^*}{\partial i} > 0$. ■

Appendix C

Properties of the Optimal Investment Strategy.

P.1. $\frac{\partial \pi^*}{\partial k} < 0$ and $\frac{\partial \pi^*}{\partial i} > 0$. Recall that function H_2 is independent of parameters (k, i) and that $u = \frac{H_2'(x)}{H_2'(x^*)}$. Fixing u, we have

$$\begin{aligned} \frac{\partial \pi^*}{\partial k} &= \frac{\partial \pi^*}{\partial x}\frac{\partial x}{\partial k} \\ &= \frac{\partial \pi^*}{\partial x}\frac{uH_2''(x^*)}{H_2''(x)}\frac{\partial x^*}{\partial k} < 0 \text{ as } \frac{\partial \pi^*}{\partial x} > 0 \text{ and } \frac{\partial x^*}{\partial k} < 0. \end{aligned}$$

Similarly

$$\frac{\partial \pi^*}{\partial i} = \frac{\partial \pi^*}{\partial x}\frac{uH_2''(x^*)}{H_2''(x)}\frac{\partial x^*}{\partial i} > 0 \text{ as } \frac{\partial \pi^*}{\partial x} > 0 \text{ and } \frac{\partial x^*}{\partial i} > 0. \blacksquare$$

P.2. For all $u \in (0, 1)$, $\frac{\partial \pi^*}{\partial u} < 0$. Recall that

$$\pi^* = -\frac{\mu - r}{\sigma^2}\frac{xH_2''(x)}{H_2'(x)}.$$

As $H_2(x) \underset{\infty}{\sim} B(-\beta_2, \beta_1 + 1)x^{\beta_2}$, we find that $\frac{xH_2''(x)}{H_2'(x)} \underset{\infty}{\sim} \beta_2 - 1$. Then, using the expressions found for functions H_2' and H_2'', we obtain that

$$\begin{aligned} -\frac{xH_2''(x)}{H_2'(x)} &= (\beta_1 - 1)\frac{\int_0^1 e^{-\frac{\Lambda t}{x}}t^{1-\beta_2}(1-t)^{\beta_1-2}dt}{\int_0^1 e^{-\frac{\Lambda t}{x}}t^{-\beta_2}(1-t)^{\beta_1-1}dt} \\ &= (\beta_1 - 1)(-1 + \frac{\int_0^1 e^{-\frac{\Lambda t}{x}}t^{-\beta_2}(1-t)^{\beta_1-2}dt}{\int_0^1 e^{-\frac{\Lambda t}{x}}t^{-\beta_2}(1-t)^{\beta_1-1}dt}). \end{aligned}$$

Let define two auxiliary functions φ and ψ with

$$\begin{aligned} \varphi(x) &= \int_0^1 e^{-\frac{\Lambda t}{x}}t^{-\beta_2}(1-t)^{\beta_1-2}dt \\ \psi(x) &= \int_0^1 e^{-\frac{\Lambda t}{x}}t^{-\beta_2}(1-t)^{\beta_1-1}dt, \end{aligned}$$

so that

$$\begin{aligned} \varphi'(x) &= \frac{\Lambda}{x^2}\int_0^1 e^{-\frac{\Lambda t}{x}}t^{1-\beta_2}(1-t)^{\beta_1-2}dt \\ \psi'(x) &= \frac{\Lambda}{x^2}\int_0^1 e^{-\frac{\Lambda t}{x}}t^{1-\beta_2}(1-t)^{\beta_1-1}dt, \end{aligned}$$

and let $\varpi(x) = \frac{\varphi(x)}{\psi(x)}$. It follows that

$$\frac{d\varpi(x)}{dx} = \frac{\varphi'(x)\psi(x) - \psi'(x)\varphi(x)}{\psi^2(x)}.$$

Next observe that

$$\begin{aligned}\psi(x) &= \frac{\Lambda}{x^2}(\int_0^1 e^{-\frac{\Lambda t}{x}}t^{-\beta_2}(1-t)^{\beta_1-2}dt - \int_0^1 e^{-\frac{\Lambda t}{x}}t^{1-\beta_2}(1-t)^{\beta_1-2}dt)\\ \psi'(x) &= \frac{\Lambda}{x^2}(\int_0^1 e^{-\frac{\Lambda t}{x}}t^{1-\beta_2}(1-t)^{\beta_1-2}dt - \int_0^1 e^{-\frac{\Lambda t}{x}}t^{2-\beta_2}(1-t)^{\beta_1-2}dt).\end{aligned}$$

Thus $\frac{d\varpi(x)}{dx}$ has the same sign as

$$\varphi'(x)\psi(x)-\psi'(x)\varphi(x) = -\left(\int_0^1 e^{-\frac{\Lambda t}{x}}t^{1-\beta_2}(1-t)^{\beta_1-2}dt\right)^2 + \int_0^1 e^{-\frac{\Lambda t}{x}}t^{2-\beta_2}(1-t)^{\beta_1-2}dt)\times\int_0^1 e^{-\frac{\Lambda t}{x}}t^{-\beta_2}(1-t)^{\beta_1-2}dt.$$

Finally, define

$$\begin{aligned}f(t,x) &= e^{-\frac{\Lambda t}{2x}}t^{1-\frac{\beta_2}{2}}(1-t)^{\frac{\beta_1}{2}-1}\\ g(t,x) &= e^{-\frac{\Lambda t}{2x}}t^{-\frac{\beta_2}{2}}(1-t)^{\frac{\beta_1}{2}-1},\end{aligned}$$

so that $\frac{d\varpi(x)}{dx}$ has the same sign as

$$-\left(\int_0^1 f(t,x)g(t,x)dt\right)^2 + \left(\int_0^1 f^2(t,x)dt\right)\times\left(\int_0^1 g^2(t,x)dt\right) > 0 \text{ (Cauchy-Schwartz inequality).}$$

We conclude that $\frac{d\varpi(x)}{dx} > 0$ for all x. It follows that

$$\frac{d\pi^*}{du} = \frac{\partial\pi^*}{dx}\times\frac{\partial x}{\partial u} < 0 \text{ as } \frac{\partial u}{\partial x} = -J''(x) < 0.$$

We deduce that for all $u \in [0,1]$, $-\frac{f'(u)}{uf''(u)} \leq 1-\beta_2$. Then, integrating this relationship and using the fact that $f'(u) \underset{0}{\sim} K_0 u^{-\frac{1}{1-\beta_2}}$, we find that for all $u \in [0,1]$, $f'(u) \leq K_0 u^{-\frac{1}{1-\beta_2}}$, which implies that $f(u) \leq \frac{\beta_2-1}{\beta_2}K_0 u^{\frac{\beta_2}{\beta_2-1}}$. ■

Properties of Value Function F.

P1: $F_2 > 0$, $F_{22} < 0$. $F_2(W,M) = f(u) - uf'(u) = J(x) > 0$. Then $F_{22}(W,M) = \frac{u^2}{M}f''(u) < 0$.

P2: $\frac{\partial f(u)}{\partial c_F} < 0$. Let $c_{F_2} > c_{F_1}$ given. Let F^i denote the value function that corresponds to parameter c_{F_i}, $i = 1,2$. Using the HJB satisfied by F^1, it is easy to check that F^1 satisfies:

$$\begin{aligned}F^1(W_0,M_0) &= \max_{\pi} E_0\left[\int_0^{\tau_0\wedge\infty} e^{-(\theta+\delta)t}\left[(c_{F_2}-c_{F_1})W_t\left[(f^1)'(u_t)-1\right]dt + c_{F_2}W_t dt + kdM_t)\right]\right]\\ \text{s.t. } dW_t &= (r-c_{F_2})W_t dt + (\mu-r)\pi_t W_t dt + \sigma\pi_t W_t dw_t - (k-i)dM_t.\end{aligned}$$

Recall we established that $(f^1)'(u_t) = x_t > 1$. As $(c_{F_2}-c_{F_1})W_t\left[(f^1)'(u_t)-1\right] > 0$, we deduce that $F^1 > F^2$. Finally, since $(1+i)f(1) = (1+k)x^* - 1$, we deduce that $\frac{\partial x^*}{\partial c_F} = \frac{1+i}{1+k}\frac{\partial f(1)}{\partial c_F} < 0$. ■

P3: $\frac{\partial f(u)}{\partial k} < 0$ and $\frac{\partial f(u)}{\partial i} > 0$. Recall that

$$f(u) = -\frac{H_2(x) - xH_2'(x)}{H_2'(x^*)} \text{ and } u = -\frac{H_2'(x)}{H_2'(x^*)}.$$

Fixing $u \geq 0$, we have

$$\begin{aligned}
\frac{\partial f(u)}{\partial k} &= \frac{[H_2(x) - xH_2'(x)]\, H_2''(x^*)}{[H_2'(x^*)]^2}\frac{\partial x^*}{\partial k} + \frac{xH_2''(x)}{H_2'(x^*)}\frac{\partial x}{\partial k} \\
-uH_2''(x^*)\frac{\partial x^*}{\partial k} &= H_2''(x)\frac{\partial x}{\partial k}.
\end{aligned}$$

Rearranging terms and simplifying, we find that $\frac{\partial f(u)}{\partial k} = \frac{H_2(x)H_2''(x^*)}{\left[H_2'(x^*)\right]^2}\frac{\partial x^*}{\partial k} < 0$. Similarly, $\frac{\partial f(u)}{\partial i} = \frac{H_2(x)H_2''(x^*)}{\left[H_2'(x^*)\right]^2}\frac{\partial x^*}{\partial i} > 0$. ■

Process x**.** Recall that for $u_t < 1$,

$$du_t = \left[(r - c_F)u_t - \frac{(\mu - r)^2}{\sigma^2}\frac{f'(u_t)}{f''(u_t)}\right] dt - \frac{\mu - r}{\sigma}\frac{f'(u_t)}{f''(u_t)}dw_t.$$

Since $u_t = -J'(x_t)$, formally writing

$$dx_t = \mu_{x_t}dt + \sigma_{x_t}dw_t,$$

and applying Ito's lemma for $x_t > x^*$ leads to

$$du_t = -\left(J''(x_t)\mu_{x_t} + \frac{1}{2}J'''(x_t)\sigma_{x_t}^2\right) dt - J''(x_t)\sigma_{x_t}dw_t.$$

Identifying the drift and the diffusion terms from the budget constraint, and using the fact that $x_t = f'(u_t)$, $u_t = -J'(x_t)$ and $f''(u_t) = -\frac{1}{J''(x_t)}$, we find that

$$\begin{aligned}
\sigma_{x_t} &= -\frac{\mu - r}{\sigma}x_t \\
-(r - c_F)J'(x_t) + \frac{(\mu - r)^2}{\sigma^2}x_tJ''(x_t) &= -J''(x_t)\mu_{x_t} - \frac{\sigma_{x_t}^2}{2}J'''(x_t).
\end{aligned}$$

Then, recall that function J is solution of (6), so differentiating both sides of (6) and rearranging terms yields

$$(r - c_F)J'(x) = [(\theta + \delta - r + c_F)x - c_F)]J''(x) + \frac{(\mu - r)^2}{\sigma^2}xJ''(x) + \frac{1}{2}\frac{(\mu - r)^2}{\sigma^2}x^2J'''(x).$$

Given what precedes, this implies that we must have

$$\mu_{x_t} = (\theta + \delta - r + c_F)x - c_F. \blacksquare$$

$\frac{\partial \pi^*}{\partial k} < 0$ and $\frac{\partial \pi^*}{\partial i} > 0$. Recall that function H_2 is independent of parameters (k, i) and that $u = \frac{H_2'(x)}{H_2'(x^*)}$. Fixing u, we have

$$\begin{aligned} \frac{\partial \pi^*}{\partial k} &= \frac{\partial \pi^*}{\partial x}\frac{\partial x}{\partial k} \\ &= \frac{\partial \pi^*}{\partial x}\frac{uH_2''(x^*)}{H_2''(x)}\frac{\partial x^*}{\partial k} < 0 \text{ as } \frac{\partial \pi^*}{\partial x} > 0 \text{ and } \frac{\partial x^*}{\partial k} < 0. \end{aligned}$$

Similarly

$$\frac{\partial \pi^*}{\partial i} = \frac{\partial \pi^*}{\partial x}\frac{uH_2''(x^*)}{H_2''(x)}\frac{\partial x^*}{\partial i} > 0 \text{ as } \frac{\partial \pi^*}{\partial x} > 0 \text{ and } \frac{\partial x^*}{\partial i} > 0. \ \blacksquare$$

Appendix D

Appendix D1

Revenue Decomposition. Recall that for $x > x^*$ functions g_k satisfies the following ODE

$$(\theta + \delta)g_k(x) = [(\theta + \delta - r + c_F)x - c_F)]g_k'(x) + \frac{1}{2}\frac{(\mu - r)^2}{\sigma^2}x^2 g_k''(x), \tag{19}$$

with $\lim_{\infty} g_k = 0$. The solution of (19) that vanishes when x goes to ∞ is given by

$$g_k(x) = -\frac{A_k H_2(x)}{H_2'(x^*)},$$

where $A_k > 0$ is a constant to be determined and recall that $-\frac{H_2(x)}{H_2'(x^*)} = f(u) - uf'(u)$. Furthermore

$$\begin{aligned} f_k(1) &= g_k(x^*) = -\frac{A_k H_2(x^*)}{H_2'(x^*)} \\ f_k'(1) &= -\frac{g_k'(x^*)}{J''(x^*)} = -\frac{A_k H_2'(x^*)}{H_2''(x^*)}. \end{aligned}$$

As $(1 + k)f_k'(1) = k + (1 + i)f_k(1)$, solving for constant A_k leads to

$$A_k = \frac{k}{-(k+1)\frac{H_2'(x^*)}{H_2''(x^*)} + (1+i)\frac{H_2(x^*)}{H_2'(x^*)}}.$$

To show that indeed $A_k > 0$, recall that

$$\varphi_2(x) = k(ax - 1)\frac{H_2'(x)}{H_2(x)} + 1 + i,$$

with $\varphi_2(x^*) = 0$ and $\varphi_2'(x^*) < 0$. We have

$$\varphi_2'(x) = ka\frac{H_2'(x)}{H_2(x)} + k(ax-1)\frac{H_2''(x)}{H_2'(x)} - k(ax-1)\frac{H_2'(x)}{H_2(x)}\frac{H_2'(x)}{H_2(x)}.$$

Then, using the fact that $\varphi_2(x^*) = 0$ leads to

$$\begin{aligned}\varphi_2'(x^*) &= ka\frac{H_2'(x^*)}{H_2(x^*)} - (1+i)\frac{H_2''(x^*)}{H_2'(x^*)} + (1+i)\frac{H_2'(x^*)}{H_2(x^*)} \\ &= -\frac{H_2''(x^*)}{H_2(x^*)}\left[-(k+1)\frac{H_2'(x^*)}{H_2''(x^*)} + (1+i)\frac{H_2(x^*)}{H_2'(x^*)}\right], \text{ as } ka+1+i = k+1,\end{aligned}$$

which indeed implies that $A_k > 0$ as $\varphi_2'(x^*) < 0$ and $-\frac{H_2''(x^*)}{H_2(x^*)} < 0$. Next, observe that

$$f_k'(u) = -\frac{A_k H_2'(x)}{H_2''(x)} = -A_k u f''(u) > 0.$$

Finally, as π^* is decreasing in u, we have $1 < \frac{f'(u)}{u(f''(u))^2}(f''(u) + uf'''(u))$, which implies that $f_k'' < 0$. Finally

$$\frac{f_k(u)}{f(u)} = A_k(1 - \frac{uf'(u)}{f(u)}).$$

Using relationship (17), we find that $\lim_{0^-}\frac{f_k(u)}{f(u)} = \frac{A_k}{1-\beta_2}$, so in particular $A_k < 1-\beta_2$. Furthermore, observe that

$$\frac{f(u)}{uf'(u)} = -\frac{H_2(x)}{xH_2'(x)} + 1.$$

Following the same steps as for π^* as for P.2. in Appendix C, one can show that function λ with $\lambda(x) = -\frac{xH_2'(x)}{H_2(x)}$ is increasing in x. Since, $\frac{f_k(u)}{f(u)} = \frac{A_k}{1+\lambda(x)}$, we deduce that $\frac{\partial}{\partial u}\left[\frac{f_k(u)}{f(u)}\right] = \frac{\partial}{\partial x}\left[\frac{f_k(u)}{f(u)}\right] \times \frac{\partial x}{\partial u} > 0$ as $\frac{\partial}{\partial x}\left[\frac{f_k(u)}{f(u)}\right] < 0$ and $\frac{\partial x}{\partial u} < 0$. ■

Appendix D2

Expected Time until next HWM Hit. Let $f(x,a) = E[e^{-a\tau}]$ be the Laplace transform of the hitting time τ, for $x_0 = x > x^*$ and $a \geq 0$. Set $a_0 = \theta + \delta - r + c_F$, $a_1 = c_F$ and $a_2 = \frac{1}{2}\frac{(\mu-r)^2}{\sigma^2}$, so that $\frac{a_0}{a_2} = 1 - \beta_1 - \beta_2$ and $\frac{a_1}{a_2} = \Lambda$. For $x > x^*$, function f is a smooth function that satisfies the following ODE for

$$af(x,a) = (a_0 x - a_1)f_1(x,a) + a_2 x^2 f_{11}(x,a),$$

with $f(x^*,a) = 1$ and $\lim_{x\to\infty} f(x,a) = 0$. The solution is given by

$$f(x,a) = \Big(\frac{x}{x^*}\Big)^{\beta_{2,a}} \frac{\int_0^1 e^{-\frac{\Lambda t}{x}} t^{-\beta_{2,a}-1}(1-t)^{\beta_{1,a}}dt}{\int_0^1 e^{-\frac{\Lambda t}{x^*}} t^{-\beta_{2,a}-1}(1-t)^{\beta_{1,a}}dt},$$

where $\beta_{1,a}$ and $\beta_{2,a}$ are respectively the positive and negative roots of the quadratic Q_a with

$$Q_a(y) = a_2 y^2 + (a_0 - a_2)y - a.$$

Observe that as a goes to 0, we have

$$\beta_{1,a} \underset{0}{\sim} \beta_1 + \beta_2 - \frac{a}{a_0 - a_2} \text{ and } \beta_{2,a} \underset{0}{\sim} \frac{a}{a_0 - a_2}.$$

It follows that $E[\tau] = -\frac{\partial f(x,a)}{\partial a}_{|a=0}$. Note that

$$\beta_{2,a} \int_0^1 e^{-\frac{\Lambda t}{x}} t^{-\beta_{2,a}-1}(1-t)^{\beta_{1,a}} dt = -\frac{\Lambda}{x}\int_0^1 e^{-\frac{\Lambda t}{x}} t^{-\beta_{2,a}}(1-t)^{\beta_{1,a}+1} dt - (\beta_{1,a} - \beta_{2,a} + 1)\int_0^1 e^{-\frac{\Lambda t}{x}} t^{-\beta_{2,a}}(1-t)^{\beta_{1,a}} dt.$$

We are now ready to take a Taylor expansion of order 1 in variable a around $a = 0$.

$$\begin{aligned}
& -\frac{\Lambda}{x}\int_0^1 e^{-\frac{\Lambda t}{x}} t^{-\beta_{2,a}}(1-t)^{\beta_{1,a}+1} dt - (\beta_{1,a} - \beta_{2,a} + 1)\int_0^1 e^{-\frac{\Lambda t}{x}} t^{-\beta_{2,a}}(1-t)^{\beta_{1,a}} dt \\
= & -\frac{\Lambda}{x}\int_0^1 e^{-\frac{\Lambda t}{x}} (1-t)^{2-\frac{a_0}{a_2}} (1 - \frac{a}{a_0 - a_2}\ln[t(1-t)]) dt \\
& -(1 - \frac{a_0}{a_2} - \frac{2a}{a_0 - a_2}) \int_0^1 e^{-\frac{\Lambda_1 t}{x}} (1-t)^{1-\frac{a_0}{a_2}} \left(1 - \frac{a}{a_0 - a_2}\ln[t(1-t)]\right) dt.
\end{aligned}$$

The term of order 0 is given by

$$-\int_0^1 e^{-\frac{\Lambda t}{x}} (1-t)^{\beta_1+\beta_2} \left[\frac{\Lambda(1-t)}{x} + 1 + \beta_1 + \beta_2\right] dt = -1,$$

whereas the term of order 1 is given by

$$\frac{a}{a_0 - a_2} \times \int_0^1 e^{-\frac{\Lambda t}{x}} (1-t)^{\beta_1+\beta_2} \left[\left(\frac{\Lambda(1-t)}{x} + 1 + \beta_1 + \beta_2\right)\ln[t(1-t)] + 2\right] dt.$$

Let denote

$$\begin{aligned}
A(x) &= \int_0^1 e^{-\frac{\Lambda t}{x}} (1-t)^{\beta_1+\beta_2} \left[1 + \left(\frac{\Lambda(1-t)}{x} + \beta_1 + \beta_2 + 1\right)\ln t\right] dt. \\
&= \int_0^1 e^{-\frac{a_1 t}{a_2 x}} (1-t)^{1-\frac{a_0}{a_2}} \left[1 + \left(\frac{a_1(1-t)}{a_2 x} + 2 - \frac{a_0}{a_2}\right)\ln t\right] dt.
\end{aligned}$$

It follows that

$$\begin{aligned}
f(x,a) &= \left(1 + \frac{a}{a_0 - a_2}\ln\frac{x}{x^*}\right)\left(\frac{-1 + \frac{a}{a_0 - a_2}A(x)}{-1 + \frac{a}{a_0 - a_2}A(x^*)}\right) + o(a) \\
&= 1 + \frac{a}{a_0 - a_2}\left(\ln\frac{x}{x^*} + A(x^*) - A(x)\right) + o(a).
\end{aligned}$$

We deduce that

$$E[\tau] = \frac{1}{(\beta_1+\beta_2)\frac{1}{2}\frac{(\mu-r)^2}{\sigma^2}}\left(\ln\frac{x}{x^*} + A(x^*) - A(x)\right). \blacksquare$$

Appendix E: Large Drawdown Prohibited

Appendix E1: No Management Fee $c_F = 0$

Proof of proposition 3. Set $x_\alpha^{**} = f'(\alpha)$ and $x_\alpha^* = f'(1)$. We want to show existence and uniqueness of the following system S

$$\begin{aligned}
\alpha &= -\beta_1 K_1 (x_\alpha^{**})^{\beta_1-1} - \beta_2 K_2 (x_\alpha^{**})^{\beta_2-1} &(20)\\
0 &= \beta_1(\beta_1-1)K_1(x_\alpha^{**})^{\beta_1-1} + \beta_2(\beta_2-1)K_2(x_\alpha^{**})^{\beta_2-1} &(21)\\
1 &= -\beta_1 K_1 (x_\alpha^{*})^{\beta_1-1} - \beta_2 K_2 (x_\alpha^{*})^{\beta_2-1} &(22)\\
\frac{k-i}{1+i}x_\alpha^* &= \frac{k}{1+i} + K_1(x_\alpha^*)^{\beta_1} + K_2(x_\alpha^*)^{\beta_2}, &(23)
\end{aligned}$$

with $0 < x_\alpha^* < x_\alpha^{**}$. Combining relationships (20) and (21) leads to

$$\begin{aligned}
K_1(x_\alpha^{**})^{\beta_1-1} &= -\frac{1-\beta_2}{\beta_1(\beta_1-\beta_2)}\alpha < 0\\
K_2(x_\alpha^{**})^{\beta_2-1} &= -\frac{\beta_1-1}{\beta_2(\beta_1-\beta_2)}\alpha > 0.
\end{aligned}$$

Then, combining relationships (22) and (23) leads to

$$\begin{aligned}
K_1(x_\alpha^*)^{\beta_1} &= \frac{1}{\beta_1-\beta_2}\left[-(\beta_2\frac{k-i}{1+i}+1)x_\alpha^* + \frac{\beta_2 k}{1+i}\right]\\
K_2(x_\alpha^*)^{\beta_2} &= \frac{1}{\beta_1-\beta_2}\left[(\beta_1\frac{k-i}{1+i}+1)x_\alpha^* - \frac{\beta_1 k}{1+i}\right].
\end{aligned}$$

Then set $\varpi = \frac{x_\alpha^*}{x_\alpha^{**}} < 1$. Eliminating constants K_1 and K_2, we find that

$$\begin{aligned}
-\frac{1-\beta_2}{\beta_1}\alpha x_\alpha^* \varpi^{\beta_1-1} &= -(\beta_2\frac{k-i}{1+i}+1)x_\alpha^* + \frac{\beta_2 k}{1+i}\\
-\frac{\beta_1-1}{\beta_2}\alpha x_\alpha^* \varpi^{\beta_2-1} &= (\beta_1\frac{k-i}{1+i}+1)x_\alpha^* - \frac{\beta_1 k}{1+i}.
\end{aligned}$$

Eliminating x_α^* yields

$$\alpha[(1-\beta_2)\varpi^{\beta_1-1} + (\beta_1-1)\varpi^{\beta_2-1}] = \beta_1 - \beta_2. \quad (24)$$

For $z \in (0,1]$, define auxiliary function Φ with

$$\Phi(z) = \alpha\left((1-\beta_2)z^{\beta_1-1} + (\beta_1-1)z^{\beta_2-1}\right) - (\beta_1-\beta_2).$$

Φ is a continuously differentiable function with

$$\Phi'(z) = \alpha(1-\beta_2)(\beta_1-1)z^{\beta_2-2}\left(z^{\beta_1-\beta_2}-1\right) < 0 \text{ for all } z \in (0,1).$$

Hence, Φ is strictly decreasing with $\lim_{0^+}\Phi = \infty$ and $\Phi(1) = (\alpha-1)(\beta_1-\beta_2) < 0$. We conclude that Φ has a unique root ϖ in $(0,1)$ than is independent of k. Furthermore, totally differentiating relationship (24) with respect to α leads to

$$(1-\beta_2)\varpi^{\beta_1-1} + (\beta_1-1)\varpi^{\beta_2-1} + \Phi'(\varpi)\frac{\partial\varpi}{\partial\alpha} = 0,$$

so we can conclude that $\frac{\partial\varpi}{\partial\alpha} > 0$. We still need to check the condition $x^*_\alpha > 0$ or equivalently $\frac{k-i}{1+i} + \frac{1}{\beta_2} - \frac{1-\beta_2}{\beta_1\beta_2}\alpha\varpi^{\beta_1-1} > 0$, i.e.,

$$\beta_1((k-i)\beta_2 + 1 + i) - \alpha(1-\beta_2)(1+i)\varpi^{\beta_1-1} < 0.$$

Set $\varpi^* = \left[\frac{\beta_1((k-i)\beta_2+1+i)}{\alpha(1+i)(1-\beta_2)} \vee 0\right]^{\frac{1}{\beta_1-1}} \geq 0$. It is easy to verify that above condition is met whenever $\varpi > \varpi^*$, or equivalently $\Phi(\varpi^* \wedge 1) > 0$. We find that

$$x^*_\alpha = \frac{\beta_2\beta_1 k}{\beta_1((k-i)\beta_2+1+i) - \alpha(1-\beta_2)(1+i)\varpi^{\beta_1-1}}.$$

At $u=1$, we have $(1+k)f'(1) = k + (1+i)f(1)$ and recall that $x^*_\alpha = f'(1)$. As clearly $\frac{\partial f(1)}{\partial\alpha} < 0$, we obtain that $\frac{\partial x^*_\alpha}{\partial\alpha} < 0$. Then, as $x^{**}_\alpha = \frac{x^*_\alpha}{\varpi}$, we get $\frac{\partial x^{**}_\alpha}{\partial\alpha} < 0$ and, we can recover constants K_1 and K_2. Since $K_1 = -\frac{1-\beta_2}{\beta_1(\beta_1-\beta_2)}\alpha(x^{**}_\alpha)^{1-\beta_1}$, we deduce that $\frac{\partial K_1}{\partial\alpha} < 0$. Finally observe that

$$\begin{aligned}\pi^* &= -\frac{\mu-r}{\sigma^2}\frac{\beta_1(\beta_1-1)K_1x^{\beta_1-1} + \beta_2(\beta_2-1)K_2x^{\beta_2-1}}{\beta_1K_1x^{\beta_1-1} + \beta_2K_2x^{\beta_2-1}} \\ &= -\frac{\mu-r}{\sigma^2}\left[\beta_1 - 1 - (\beta_1-\beta_2)\frac{\beta_2K_2}{\beta_1K_1(f'(u))^{\beta_1-\beta_2} + \beta_2K_2}\right],\end{aligned}$$

which is increasing in u as $f' > 0$, $K_1 < 0$ and $K_2 > 0$. ■

$\frac{\partial\pi^*}{\partial\alpha} < 0$. Recall that $u = -\beta_1K_1x^{\beta_1-1} - \beta_2K_2x^{\beta_2-1}$, so, fixing u, we have

$$\frac{1}{x}\frac{\partial x}{\partial\alpha} = -\frac{\beta_1x^{\beta_1-1}\frac{\partial K_1}{\partial\alpha} + \beta_2x^{\beta_2-1}\frac{\partial K_2}{\partial\alpha}}{\beta_1(\beta_1-1)K_1x^{\beta_1-1} + \beta_2(\beta_2-1)K_2x^{\beta_2-1}}.$$

Since $\pi^* = -\frac{\mu-r}{\sigma^2}\frac{f'(u)}{uf''(u)} = \frac{\mu-r}{\sigma^2 u}\left(\beta_1(\beta_1-1)K_1x^{\beta_1-1}+\beta_2(\beta_2-1)K_2x^{\beta_2-1}\right)$, it follows that

$$\begin{aligned}\frac{\partial\pi^*}{\partial\alpha} &= -\frac{\mu-r}{\sigma^2 u}\Bigg(\beta_1(\beta_1-1)\frac{\partial K_1}{\partial\alpha}x^{\beta_1-1}+\beta_2(\beta_2-1)\frac{\partial K_2}{\partial\alpha}x^{\beta_2-1}\\ &\quad+\left(\beta_1(\beta_1-1)^2K_1x^{\beta_1-1}+\beta_2(\beta_2-1)^2K_2x^{\beta_2-1}\right)\Bigg)\frac{1}{x}\frac{\partial x}{\partial\alpha}\\ &= -\frac{(\mu-r)x^{\beta_1+\beta_2-1}}{\sigma^2 u}\frac{\beta_1\beta_2(\beta_1-\beta_2)((\beta_2-1)K_2\frac{\partial K_1}{\partial\alpha}-(\beta_1-1)K_1\frac{\partial K_2}{\partial\alpha})}{-\beta_1(\beta_1-1)K_1x^{\beta_1}-\beta_2(\beta_2-1)K_2x^{\beta_2}}.\end{aligned}$$

Since $-\beta_1(\beta_1-1)K_1x^{\beta_1}-\beta_2(\beta_2-1)K_2x^{\beta_2} = -x^2J''(x) < 0$, we conclude that $\frac{\partial\pi^*}{\partial\alpha}$ has a constant sign, independent of u. Then

$$\begin{aligned}\pi_1^* &= \frac{\mu-r}{\sigma^2}\left(\beta_1(\beta_1-1)K_1(x_\alpha^*)^{\beta_1-1}+\beta_2(\beta_2-1)K_2(x_\alpha^*)^{\beta_2-1}\right)\\ &= -\frac{\mu-r}{\sigma^2x_\alpha^*}\frac{1}{\beta_1-\beta_2}\left(\beta_1(\beta_1-1)\left[-(\beta_2\frac{k-i}{1+i}+1)x_\alpha^*+\frac{\beta_2k}{1+i}\right]+\beta_2(\beta_2-1)\left[\beta_1(\frac{k-i}{1+i}+1)x_\alpha^*-\frac{\beta_1k}{1+i}\right]\right)\\ &= \frac{\mu-r}{\sigma^2}\beta_1\beta_2\left(\frac{k}{1+i}\frac{1}{x_\alpha^*}-\frac{k-i}{1+i}-\frac{\beta_1+\beta_2-1}{\beta_1\beta_2}\right).\end{aligned}$$

It follows that $\frac{\partial\pi_1^*}{\partial\alpha} = -\frac{\mu-r}{\sigma^2}\beta_1\beta_2\frac{k}{1+i}\frac{1}{(x_\alpha^*)^2}\frac{\partial x_\alpha^*}{\partial\alpha} < 0$, as $\frac{\partial x_\alpha^*}{\partial\alpha} < 0$. The desired results follows. ■

Imposing $f_\alpha(\alpha)=0$ instead of $\pi^*(\alpha)=0$. This is the condition imposed in Lan, Wang and Yang (2012). Condition (21) is now replaced by $J_\alpha(x_\alpha^{**})+\alpha x_\alpha^{**}=0$. We find that

$$\begin{aligned}K_1(x_\alpha^{**})^{\beta_1-1} &= -\frac{1-\beta_2}{\beta_1-\beta_2}\alpha < 0\\ K_2(x_\alpha^{**})^{\beta_2-1} &= -\frac{\beta_1-1}{\beta_1-\beta_2}\alpha < 0,\end{aligned}$$

and using condition (22) leads to

$$\alpha[(1-\beta_2)\beta_1\varpi^{\beta_1-1}+(\beta_1-1)\beta_2\varpi^{\beta_2-1}] = \beta_1-\beta_2,$$

where $\varpi = \frac{x_\alpha^*}{x_\alpha^{**}}$. For $z>0$, define auxiliary function Ψ with

$$\Psi(z) = \alpha\left((1-\beta_2)\beta_1z^{\beta_1-1}+(\beta_1-1)\beta_2z^{\beta_2-1}\right)-(\beta_1-\beta_2).$$

Ψ is a continuously differentiable function with

$$\Psi'(z) = \alpha(1-\beta_2)(\beta_1-1)z^{\beta_2-2}\left(\beta_1z^{\beta_1-\beta_2}-\beta_2\right) > 0.$$

Since $\lim_0\Psi = -\infty$, $\Psi(1) = (\alpha-1)(\beta_1-\beta_2) < 0$ and $\lim_\infty\Psi = \infty$, we deduce that Ψ admits a unique root strictly greater than 1, so we must have $\varpi > 1$, i.e., $x_\alpha^* > x_\alpha^{**}$. Furthermore, since $K_1 < 0$ and $K_2 < 0$, this implies that for all x in $(x_\alpha^{**}, x_\alpha^*)$, we have $J_\alpha''(x) > 0$, i.e., for all $u > \alpha$, $f_\alpha''(u) > 0$. Value function f_α is globally convex and thus cannot be solution of equation (14). In order to have a well-defined optimization problem (4), a leverage constraint is necessary such as for instance $\pi \leq \pi_{\max}$, with $\pi_{\max} > 0$, which optimally shall always binds. ■

Appendix E2: General Case. We want to show that system (S)

$$\begin{aligned}
\alpha &= -K_1 H_1'(x_\alpha^{**}) - K_2 H_2'(x_\alpha^{**}) \\
0 &= K_1 H_1''(x_\alpha^{**}) + K_2 H_2''(x_\alpha^{**}) \\
1 &= -K_1 H_1'(x_\alpha^{*}) - K_2 H_2'(x_\alpha^{*}) \\
k(ax_\alpha^* - 1) &= K_1(1+i)H_1(x_\alpha^*) + K_2(1+i)H_2(x_\alpha^*), \text{ with } a = \frac{k-i}{k},
\end{aligned}$$

$0 \leq x_\alpha^* \leq x_\alpha^{**}$. Solving for K_1 and K_2 we find that

$$\begin{aligned}
K_1 &= \frac{(1+i)H_2(x_\alpha^*) + k(ax_\alpha^* - 1)H_2'(x_\alpha^*)}{(1+i)W(H_1, H_2)(x_\alpha^*)} \\
K_2 &= \frac{(1+i)H_1(x_\alpha^*) + k(ax_\alpha^* - 1)H_1'(x_\alpha^*)}{(1+i)W(H_2, H_1)(x_\alpha^*)} \\
K_1 &= -\frac{\alpha H_2''(x_\alpha^{**})}{W(H_1', H_2')(x_\alpha^{**})} < 0 \\
K_2 &= \frac{\alpha H_1''(x_\alpha^{**})}{W(H_1', H_2')(x_\alpha^{**})} > 0.
\end{aligned}$$

$$\frac{(1+i)H_2(x_\alpha^*) + k(ax_\alpha^* - 1)H_2'(x_\alpha^*)}{(1+i)W(H_2, H_1)(x_\alpha^*)} = \frac{\alpha H_2''(x_\alpha^{**})}{W(H_1', H_2')(x_\alpha^{**})} \tag{25}$$

$$\frac{(1+i)H_1(x_\alpha^*) + k(ax_\alpha^* - 1)H_1'(x_\alpha^*)}{(1+i)W(H_2, H_1)(x_\alpha^*)} = \frac{\alpha H_1''(x_\alpha^{**})}{W(H_1', H_2')(x_\alpha^{**})}. \tag{26}$$

$$\frac{(1+i)H_2(x_\alpha^*) + k(ax_\alpha^* - 1)H_2'(x_\alpha^*)}{(1+i)H_1(x_\alpha^*) + k(ax_\alpha^* - 1)H_1'(x_\alpha^*)} = \frac{H_2''(x_\alpha^{**})}{H_1''(x_\alpha^{**})} \tag{27}$$

Note that function Φ_2 where $\Phi_2(x) = (1+i)H_2(x) + k(ax-1)H_2'(x) = H_2(x)\varphi_2(x)$ is the product of two (strictly) decreasing functions that are positive on $(0, x^*)$. Then set Φ_1 where $\Phi_1(x) = (1+i)H_1(x) + k(ax-1)H_1'(x) = H_1(x)\varphi_1(x)$ with

$$\varphi_1(x) = (1+i) + k(ax-1)\frac{H_1'(x)}{H_1(x)}.$$

Following the exact same steps as in Appendix B and using the expressions for H_1 and H_1', one can show that function φ_1 is (strictly) increasing with $\lim_{x\to\infty} \varphi_1(x) = (k-i)\beta_1 + 1 + i > 0$ and $\varphi_1(x) \underset{0}{\sim} -\frac{k\Lambda}{x^2}$. We deduce that φ_1 has a unique root $x_{\min}^*$ and note that $x_{\min}^* < 1$. We conclude that Φ_1 is increasing and positive on $(x_{\min}^*, \infty)$. It follows that $\Phi = \frac{\Phi_2}{\Phi_1}$ is decreasing and positive on $(x_{\min}^*, x^*)$ that takes value in $(0, \infty)$. Then, observe that function $\frac{H_2''}{H_1''}$ is a decreasing and positive function. By the Implicit Function Theorem, we can write $x_\alpha^* = \Psi(x_\alpha^{**})$, where Ψ is a positive increasing function, independent of α, that takes value in $(x_{\min}^*, x^*)$ with $\lim_{x\to 0} \Psi(x) = x_{\min}^*$ and $\lim_{x\to\infty} \Psi(x) = x^*$. We look for a fixed point

$\overline{x}$ of Ψ. Using relationship (27) and rearranging terms, we find that $\overline{x}$ must satisfy

$$(1+i)\left[H_2(\overline{x})H_1''(\overline{x}) - H_1(\overline{x})H_2''(\overline{x})\right] = k(a\overline{x}-1)W(H_1', H_2')(\overline{x}).$$

Using the fact that function H_1 and H_2 are solutions of the (6) and the property of the Wronskian yields

$$(1+i)\left[(1-\beta_1-\beta_2)\overline{x} - \Lambda\right] = k(a\overline{x}-1)\beta_1\beta_2, \text{ with } a = \frac{k-i}{k},$$

which leads to

$$\overline{x} = \frac{\Lambda(1+i) - k\beta_1\beta_2}{(1+i)(1-\beta_1-\beta_2) - (k-i)\beta_1\beta_2}. \tag{28}$$

The condition $(k-i)\beta_2 + 1 + i < 0$ implies that the denonimator of the above fraction is positive, so indeed $\overline{x} > 0$. Therefore Ψ has a unique fixed point. Then, as Φ is decreasing and $\Phi(\overline{x}) > 0$, $\Phi(x^*) = 0$, it must be the case that $\overline{x} < x^*$. Since we are looking for $x_\alpha^* \leq x_\alpha^{**}$, we shall restrict our attention to

$$\overline{x} \leq x_\alpha^* \leq x^* \text{ and } \overline{x} \leq x_\alpha^{**}.$$

We already know that for $\alpha = 0$, the solution is $(x_\alpha^*, x_\alpha^{**}) = (x^*, \infty)$. Next, we show that for $\alpha = 1$, the solution is $x_\alpha^* = x_\alpha^{**} = \overline{x}$. It is enough to that check $x_\alpha^* = x_\alpha^{**} = \overline{x}$ satisfy relationship (25) when $\alpha = 1$, which is indeed the case. Then, manipulating relationships (26) and (25) to eliminate the term $k(ax_\alpha^* - 1)$ leads to

$$W(H_1', H_2')(x_\alpha^{**}) = \alpha\left[H_2''(x_\alpha^{**})H_1'(x_\alpha^*) - H_1''(x_\alpha^{**})H_2'(x_\alpha^*)\right].$$

Fix $x \geq \overline{x}$, and for $\overline{x} \leq y \leq x$, consider auxiliary function $G(y;x) = H_2''(x)H_1'(y) - H_1''(x)H_2'(y)$. G is a smooth function of y with

$$G'(y;x) = H_2''(x)H_1''(y) - H_1''(x)H_2''(y).$$

Notice that $G(x;x) = W(H_1', H_2')(x)$. We want to show that $G' < 0$, or equivalently that function R with $R(z) = \frac{H_2''(z)}{H_1''(z)}$ is decreasing. We have

$$\begin{aligned} R'(z) &= \frac{W(H_1'', H_2'')(z)}{\left[H_1''(z)\right]^2} \\ &= \frac{\beta_1\beta_2(\beta_1-1)(\beta_2-1)x^{-4}W(H_1, H_2)(z)}{\left[H_1''(z)\right]^2} < 0. \end{aligned}$$

It follows that, given $x \geq \overline{x}$, function Γ with $\Gamma(y;x) = \frac{G(x;x)}{G(y;x)}$ is strictly increasing for $y \leq x$ and takes value in $[0,1]$. Thus the equation $\Gamma(y;x) = \alpha$ has at most one root. Finally, consider auxiliary function

$$\begin{aligned} \Delta: \quad [\overline{x}, \infty) &\to \mathbb{R} \\ x &\mapsto \Gamma(\Psi(x); x). \end{aligned}$$

We have $\Delta(\overline{x}) = 1$ and

$$\begin{aligned}\lim_{x\to\infty}\Delta(x) &= \lim_{x\to\infty}\frac{-\beta_1\beta_2\Lambda^{\beta_2-\beta_1}\Gamma(-\beta_2)\Gamma(\beta_1+1)e^{-\frac{\Lambda}{x}}x^{\beta_1+\beta_2-3}}{H_2''(x)H_1'(x^*)-H_1''(x)H_2'(x^*)}\\ &= \lim_{x\to\infty}\frac{-\beta_1\beta_2\Lambda^{\beta_2-\beta_1}\Gamma(-\beta_2)\Gamma(\beta_1+1)e^{-\frac{\Lambda}{x}}x^{\beta_1+\beta_2-3}}{-H_2'(x^*)\beta_1(\beta_1-1)\Lambda^{\beta_2-\beta_1}x^{\beta_1-2}}\\ &= 0,\end{aligned}$$

as $\beta_2 - 1 < 0$ and where we have used the fact that $\Psi(\overline{x}) = \overline{x}$ and $\lim_{\infty}\Psi = x^*$. By the Intermediate Value Theorem, we deduce that the equation $\Delta(x) = \alpha$, with $\alpha < 1$ has (at least) one root. Given what procedes, it has at most one root, so the root is indeed unique. Finally, to show that J is strictly convex, it is enough to show that for all $x^*_\alpha < x < x^{**}_\alpha$, we have

$$K_1 H_1''(x) + K_2 H_2''(x) > 0,$$

or equivalently

$$K_1 + K_2\frac{H_2''(x)}{H_1''(x)} > 0,$$

which is indeed the case as function $\frac{H_2''}{H_1''}$ is a decreasing and $K_1 + K_2\frac{H_2''(x^{**}_\alpha)}{H_1''(x^{**}_\alpha)} = 0$. ■

Fund Manager Compensation Decomposition. Recall that $g_k'(x) = -J_\alpha''(x)f_k'(u)$ and $f_k(u) = g_k(x)$ so the boundary conditions at $x = x^*_\alpha$ and $x = x^{**}_\alpha$ are

$$\begin{aligned}A_{k_1}H_1'(x^{**}_\alpha) + A_{k_2}H_2'(x^{**}_\alpha) &= 0\\ -(1+k)\left[A_{k_1}H_1'(x^*_\alpha) + A_{k_2}H_2'(x^*_\alpha)\right] &= J_\alpha''(x^*_\alpha)\left(k + (1+i)\left[A_{k_1}H_1(x^*_\alpha) + A_{k_2}H_2(x^*_\alpha)\right]\right).\end{aligned}$$

Solving for (A_{k_1}, A_{k_2}) yields the results.

Appendix F

Table A1. Benchmark Regressions, Alternative Specifications.

The dependent variable in columns 1-6 is the change in standard deviation of monthly returns during the 6 months that follow the aniversary of each fund's inception date relative to the previous 6 months. The first three columns only consider funds with inception in January, columns four through six includes all. The dependent variable in column 7 is is the standard deviation of monthly returns during the 12 months that follow the aniversary of each fund's inception date. The sample in column 7 includes only the funds with neither incentive fee nor HWM provision. Distance to HWM is the value of the fund divided by the HWM minus one, that is, it corresponds to 0 when the fund is at the HWM and decreases as the fund moves farther away from it. All variables are winsorized at the 1% level. Standard errors are robust to heteroskedasticity and clustered at the management firm level. Significance levels: * p<0.05, ** p<0.01, *** p<0.001

	Change in Risk						Risk
Risk t-1	-0.335***	-0.343***	-0.346***	-0.356***	-0.359***	-0.358***	
	(0.018)	(0.018)	(0.018)	(0.010)	(0.010)	(0.010)	
Distance to HWM		-0.038***		-0.018***	-0.018***	0.041**	-0.042
		(0.012)		(0.006)	(0.006)	(0.020)	(0.046)
Underwater			0.009***				
			(0.002)				
Rank t-1 x 10^{-3}	-0.002**						-0.004**
	(0.001)						(0.002)
Management fee					0.004***	0.004***	
					(0.002)	(0.002)	
Incentive fee					0.001***	0.001***	
					(0.000)	(0.000)	
Distance to HWM x Management fee						-0.01	-0.023
						(0.008)	(0.023)
Distance to HWM x Incentive fee						-0.002**	0.000
						(0.001)	(0.000)
Log age (years)							0.004*
							(0.003)
Return t-1							0.054***
							(0.015)
N	8133	8133	8133	35552	35552	35552	4652
R-sqr	0.334	0.337	0.336	0.228	0.23	0.23	0.742
Strategy fixed effects	YES	YES	YES	YES	YES	YES	YES
Fund fixed effects	NO	NO	NO	NO	NO	NO	YES
Year fixed effects	YES	YES	YES	YES	YES	YES	YES

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